A Global version of Grozman's theorem

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Abstract

Let X be a manifold. The classification of all equivariant bilinear maps between tensor density modules over X has been investigated by Yu. Grozman [G1], who has provided a full classification for those which are *differential* operators. Here we investigate the same question without the hypothesis that the maps are differential operators. In our paper, the geometric context is algebraic geometry and the manifold X is the circle $\operatorname{Spec}\mathbb{C}[z, z^{-1}]$.

Our main motivation comes from the fact that such a classification is required to complete the proof of the main result of [IM]. Indeed it requires to also include the case of deformations of tensor density modules.

Contents

0 Introduction

The introduction is organized as follows. The first section is devoted to the main definitions and the statement of the Grozman Theorem. In the second section, our result is stated. In the last section, the main ideas of the proof are explained.

0.1 Grozman's Theorem:

Let X be a manifold of dimension n, let W_X be the Lie algebra of vector fields over X and let M, N and P be three tensor density modules over X. The precise meaning of tensor density module will be clarified later on and the geometric context (differential geometry, algebraic geometry, ...) is not yet precised.

In a famous paper [G1], Yu. Grozman has classified all bilinear differential operators $\pi : M \times N \to P$ which are W_X-equivariant. Since differential operators are local [P], it is enough to consider the case of the formal geometry, namely $X = \operatorname{Spec} \mathbb{C}[[z_1, \ldots, z_n]]$. The most intersting and difficult part of Grozman's theorem involves the case where $\dim X = 1$, indeed the general case follows from this case.

Therefore, we will now assume that $X = \operatorname{Spec} \mathbb{C}[[z]]$. For this manifold, the tensor density modules are the modules Ω^δ, where the parameter δ runs over $\mathbb{C}$. As a $\mathbb{C}[[z]]$-module, Ω^δ is a rank one free module whose generator is denoted by $(dz)^\delta$. The structure of W_X-module on Ω^δ is described by the following formula:

$$\xi.[f.(dz)^\delta] = (\xi.f + \delta f \mathrm{div}(\xi)).(dz)^\delta$$

for any $f \in \mathbb{C}[[z]]$ and $\xi \in W_X$, where, as usual, $\xi.f = gf'$, $\mathrm{div}(\xi) = g'$ whenever $\xi = g\dfrac{d}{dz}$ for some $g \in \mathbb{C}[[z]]$. When δ is a non-negative integer, Ω^δ is the space $(\Omega^1_X)^{\otimes\delta}$, where Ω^1_X is the space of Kälher differential of X.

The space $\oplus_\delta \Omega^\delta$ can be realized as the space of symbols of twisted pseudo-differential operators on the circle (see e.g. [IM], *twisted* means that complex powers of $\dfrac{d}{dz}$ are allowed) and therefore it carries a structure of Poisson algebra. The Poisson structure (a commutative product P and a Lie bracket B) induces two series of W_X-equivariant bilinear maps, namely the maps $P^{\delta_1,\delta_2} : \Omega^{\delta_1} \times \Omega^{\delta_2} \to \Omega^{\delta_1+\delta_2}$ and the map $B^{\delta_1,\delta_2} : \Omega^{\delta_1} \times \Omega^{\delta_2} \to \Omega^{\delta_1+\delta_2+1}$. These operators are explicitly defined by:

$P^{\delta_1,\delta_2}(f_1.(dz)^{\delta_1}, f_2.(dz)^{\delta_2}) = f_1 f_2 (dz)^{\delta_1+\delta_2}$

$B^{\delta_1,\delta_2}(f_1.(dz)^{\delta_1}, f_2.(dz)^{\delta_2}) = (\delta_2 f_1' f_2 - \delta_1 f_1 f_2')(dz)^{\delta_1+\delta_2+1}$

Moreover, the de Rham operator is a W_X-equivariant map $d : \Omega^0 \to \Omega^1$. So we can obtained additional W_X-equivariant bilinear maps between tensor density module by various compositions of B^{δ_1,δ_2} and P^{δ_1,δ_2} with d. An example is provided by the map $B^{1,\delta} \circ (d \times id) : \Omega^0 \times \Omega^\delta \to \Omega^{\delta+2}$. Following Grozman, the *classical* W_X-equivariant bilinear maps are (the linear combinations of) the maps B^{δ_1,δ_2}, P^{δ_1,δ_2}, and those obtained by various compositions with d.

Grozman discovered one additional W_X-equivariant bilinear map, namely Grozman's operator $G : \Omega^{-2/3} \times \Omega^{-2/3} \to \Omega^{5/3}$ defined by the formula:

$G(f_1.(dz)^{-2/3}, f_2.(dz)^{-2/3}) = [2(f_1''' f_2 - f_2''' f_1) + 3(f_1'' f_2' - f_1' f_2'')](dz)^{5/3}$.

With this, one can state Grozman's result:

Grozman Theorem. *Any differential W_X -equivariant bilinear map $\pi : \Omega^{\delta_1} \times \Omega^{\delta_2} \to \Omega^\gamma$ between tensor density modules is either classical, or it is a scalar multiple of the Grozman operator.*

0.2 The result of the present paper:

In this paper, a similar question is investigated, namely the determination of all W_X-equivariant bilinear maps $\pi : M \times N \to P$ between tensor density modules, without the hypothesis that π is a differential operator. Since differential operators are local, we will establish a global (=non-local) version of Grozman Theorem.

For this purpose, we will make new hypotheses. From now on, the context is the algebraic geometry, and the manifold X of investigation is the *circle*, namely $\mathbb{C}^* = \mathrm{Spec}\ \mathbb{C}[z, z^{-1}]$. Set $\mathbf{W} = W_X$. Fix two parameters $\delta, s \in \mathbb{C}$ and set $\rho_{\delta,s}(\xi) = \xi + \delta\,\mathrm{div}\xi + i_\xi \alpha_s$ for any $\xi \in \mathbf{W}$, where $\alpha_s = sz^{-1}dz$. By definition, Ω_s^δ is the $\mathbf{W}$-module whose underlying space is $\mathbb{C}[z, z^{-1}]$ and the action is given by $\rho_{\delta,s}$. To describe more naturally the action $\rho_{\delta,s}$, it is convenient to denote by the symbol $z^s(dz)^\delta$ the generator of this module, and therefore the expressions $(z^{n+s}(dz)^\delta)_{n\in\mathbb{Z}}$ form a basis of Ω_s^δ. It follows easily that Ω_s^δ and Ω_u^δ are equal if $s - u$ is an integer. Therefore, we will consider the parameter s as an element of $\mathbb{C}/\mathbb{Z}$.

We will not provide a rigorous and general definition of the *tensor density modules* (see e.g. [M2]). Just say that the *tensor density modules* considered here are the $\mathbf{W}$-modules Ω_u^δ, where (δ, s) runs over $\mathbb{C} \times \mathbb{C}/\mathbb{Z}$.

As before, there are $\mathbf{W}$-equivariant bilinear maps $P_{u_1,u_2}^{\delta_1,\delta_2} : \Omega_{u_1}^{\delta_1} \times \Omega_{u_2}^{\delta_2} \to \Omega_{u_1+u_2}^{\delta_1+\delta_2}$ and $B_{u_1,u_2}^{\delta_1,\delta_2} : \Omega_{u_1}^{\delta_1} \times \Omega_{u_2}^{\delta_2} \to \Omega_{u_1+u_2}^{\delta_1+\delta_2+1}$, as well as the de Rham differential $d : \Omega_u^0 \to \Omega_u^1$. There is also a map $\rho : \Omega_u^1 \to \Omega_u^0$, which is defined as follows. For $u \not\equiv 0 \bmod \mathbb{Z}$, the opeartor d is invertible and set $\rho = d^{-1}$. For $u \equiv 0 \bmod \mathbb{Z}$, denote by $\rho : \Omega_u^1 \to \Omega_u^0$ the composite of the residue map $\mathrm{Res} : \Omega_0^1 \to \mathbb{C}$ and the natural map $\mathbb{C} \to \Omega_0^0 = \mathbb{C}[z, z^{-1}]$. By definition, a *classical* bilinear map between tensor density modules over the circle is any linear combination of the operators $B_{u_1,u_2}^{\delta_1,\delta_2}$, $P_{u_1,u_2}^{\delta_1,\delta_2}$ and those obtained by composition with d and ρ. An example of a classical operator is $\rho \circ P : \Omega_{u_1}^\delta \times \Omega_{u_2}^{1-\delta} \to \Omega_{u_1+u_2}^0$.

Of course, the Grozman operator provies a family of non-classical operators $G_{u,v} : \Omega_u^{-2/3} \times \Omega_v^{-2/3} \to \Omega_{u+v}^{5/3}$. A *trivial operator* is a scalar multiple of the bilinear map $\Omega_0^1 \times \Omega_0^1 \to \Omega_0^0, (\alpha, \beta) \mapsto \mathrm{Res}(\alpha)\mathrm{Res}(\beta)$. There is also another non-classical $\mathbf{W}$-equivariant operator $\Theta_\infty : \Omega_0^1 \times \Omega_0^1 \to \Omega_0^0$ which satisfies:

$$d\Theta_\infty(\alpha, \beta) = \mathrm{Res}(\alpha)\beta - \mathrm{Res}(\beta)\alpha$$

for any $\alpha, \beta \in \Omega_0^1$. Indeed Θ_∞ is unique modulo a trivial operator.

Our result is the following:

Theorem: *(restricted version) Let X be the circle. Any $\mathbf{W}$-equivariant bilinear map between tensor density module is either classical, or it is a scalar multiple of $G_{u,v}$ or of Θ_∞ (modulo a trivial operator).*

In the paper, a more general version, which also involves deformations of tensor density modules, is proved. Set $L_0 = z\frac{d}{dz}$. For a $\mathbf{W}$-module M and $s \in \mathbb{C}$, set $M_s = \mathrm{Ker}(L_0 - s)$. Let $\mathcal{S}$ be the class of all $\mathbf{W}$ modules M which satisfies the following condition: there exists $u \in \mathbb{C}/\mathbb{Z}$ such that

$$M = \oplus_{s\in u} M_s \text{ and } \dim M_s = 1 \text{ for all } s \in u.$$

The $\mathbb{Z}$-coset u is called the *support of* M, and it is denoted by Supp M.

It turns out that all modules of the class $\mathcal{S}$ have been classified by Kaplansky and Santharoubane [KS])] and, except deformations of Ω_0^0 and Ω_0^1, all modules of the class $\mathcal{S}$ are tensor density modules. Our full result is the classification of all $\mathbf{W}$-equivariant bilinear maps between modules of the class $\mathcal{S}$.

0.3 About the proofs:

In order to describe the proof and the organization of the paper, it is necessary to introduce the notion of germs of bilinear maps.

For any three vector spaces M, N and P, denote by $\mathbf{B}(M \times N, P)$ the space of bilinear maps $\pi : M \times N \to P$. Assume now that M, N and P are $\mathbf{W}$-modules of the class $\mathcal{S}$. For $x \in \mathbf{R}$, set $M_{\geq x} = \oplus_{\Re s \geq x} M_s$ and $N_{\geq x} = \oplus_{\Re s \geq x} N_s$. By definition, a *germ* of bilinear map from $M \times N$ to P is an element of the space $\mathcal{G}(M \times N, P) := \lim_{x\to+\infty} \mathbf{B}(M_{\geq x} \times N_{\geq x}, P)$.

It turns out that $\mathcal{G}(M \times N, P)$ is a $\mathbf{W}$-module. Denote by $\mathbf{B}_{\mathbf{W}}(M \times N, P)$ the space of $\mathbf{W}$-equivariant bilinear maps $\pi : M \times N \to P$, by $\mathbf{B}^0_{\mathbf{W}}(M \times N, P)$ the subspace of all $\pi \in \mathbf{B}_{\mathbf{W}}(M \times N, P)$ whose germ is zero and by $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ the space of $\mathbf{W}$-equivariant germs of bilinear maps from $M \times N$ to P.

There is a short exact sequence:

$$0 \to \mathbf{B}^0_{\mathbf{W}}(M \times N, P) \to \mathbf{B}_{\mathbf{W}}(M \times N, P) \to \mathcal{G}_{\mathbf{W}}(M \times N, P).$$

The paper contains three parts

Part 1 which determines $\mathbf{B}^0_{\mathbf{W}}(M \times N, P)$, see Theorem 1,

Part 2 which determines $\mathcal{G}_{\mathbf{W}}(M \times N, P)$, see Theorem 2,

Part 3 which determines

$$\mathbf{B}_{\mathbf{W}}(M \times N, P) \to \mathcal{G}_{\mathbf{W}}(M \times N, P),$$

see Theorem 3.

Part 1 is discussed in Section 5. The map Θ_∞ is an example of a degenerate map.

Part 2 is the main difficulty of the paper. One checks that $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) \leq 2$. So it is enough to determine when $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ is non-zero and when $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) = 2$. We will now explain our approach.

The *degree* of the modules of the class $\mathcal{S}$ is a multivalued function defined as follows. If $M = \Omega_s^\delta$ for some $\delta \neq 0$ or 1, set $\deg M = \delta$. Otherwise, set $\deg M = \{0, 1\}$. Next, let M, N and $P \in \mathcal{S}$ with $\delta_1 \in \deg M$, $\delta_2 \in \deg N$ and $\gamma \in \deg P$. We can assume that Supp P = Supp M + Supp N, since otherwise $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ would be obviously zero.

We introduce a 6 by 6 matrix $\mathbf{M} = (m_{i,j}(\delta_1, \delta_2, \gamma, x, y))_{1 \leq i,j \leq 6}$ whose entries are quadratic polynomials in the five variables $\delta_1, \delta_2, \gamma, x, y$ and which satisfies the following property:

$$\det \mathbf{M} = 0 \text{ for all } x, y \text{ if } \mathcal{G}_{\mathbf{W}}(M \times N, P) \neq 0.$$

Set $\det \mathbf{M} = \sum_{i,j} p_{i,j}(\delta_1, \delta_2, \gamma) x^i y^j$ and let $\mathfrak{Z}$ be the common set of zeroes of all polynomials $p_{i,j}$. Half of the entries of $\mathbf{M}$ are zero and only 16 of the 720 diagonals of $\mathbf{M}$ give a non-zero contribution for $\det \mathbf{M}$. However a human computation looks too complicated, because each non-zero entry of $\mathbf{M}$ is a linear combination of 9 or 10 distinct monomials. The computation of the polynomials $p_{i,j}$ has been done with MAPLE.

As expected, $p_{1,3}$ and $p_{3,1}$ are degree eight polynomials. It turns out that each of them is a product of 6 degree one factors and one quadratic factor. Indeed 4 degree one factors are obvious and the rest of the factorizations look miraculous. Moreover the two (suitably normalized) quadratic factors differ by a linear term. It follows that the common zero set of $p_{1,3}$ and $p_{3,1}$ is a union of affine planes, affine lines and some planar quadrics. This allows to explicitely solve the equations $p_{i,j} = 0$. Since only the polynomials $p_{1,3}, p_{3,1}$ and $p_{2,2}$ are needed, the other polynomials $p_{i,j}$ are listed in Appendix A. It turns out that $\mathfrak{Z}$ decomposes into four planes, eight lines and four points.

Using an additional trick, we determine when $\mathcal{G}_{\mathbf{W}}(M \times N, P) \neq 0$, and when $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) = 2$. Although its proof is the main difficulty of the paper, the statement of Theorem 2 is very simple. Indeed $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ is non-zero exactly when $(\delta_1, \delta_2, \gamma)$ belongs to an explicit algebraic subset $\mathfrak{z}$ of $\mathfrak{Z}$ consisting of two planes, six lines and five points. Moreover, it has dimension two iff $\{\delta_1, \delta_2, \gamma\} \subset \{0, 1\}$).

Theorem 3 determines which germs in $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ can be lifted to a $\mathbf{W}$-equivariant bilinear map. Each particular case is easy, but the list is very long. Therefore Theorem 3 has been split into Theorem 3.1 and Theorem 3.2, corresponding to the case where $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ has dimension one or two.

It should be noted that all $\mathbf{W}$-modules of the class $\mathcal{S}$ are indecomposable except one, namely $\overline{A} \oplus \mathbb{C}$, where $\overline{A} = \mathbb{C}[z, z^{-1}]/\mathbb{C}$. In most statements about bilinear maps $\pi : M \times N \to P$, we assume that M, N and P are indecomposable. Indeed the case where some modules are decomposable follows easily. The indecomposability hypothesis removes many less interesting cases. This is helpful since some statements already contain many particular cases, e.g. Theorem 3.1 contains 16 of them.

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1 The Kaplansky-Santharoubane Theorem

The *Witt algebra* $\mathbf{W}$ is the Lie algebra of derivations of the Laurent polynomial ring $A = \mathbb{C}[z, z^{-1}]$. Clearly the elements $L_n = z^{n+1}\dfrac{d}{dz}$, where n runs over $\mathbb{Z}$, form a basis of $\mathbf{W}$ and we have

$$[L_m, L_n] = (n-m)L_{m+n}.$$

Throughout this paper, $\mathfrak{sl}(2)$ refers to its subalgebra

$$\mathbb{C}L_{-1} \oplus \mathbb{C}L_0 \oplus \mathbb{C}L_1.$$

1.1 Statement of the theorem

For a $\mathbf{W}$-module M, set $M_z = \{m \in M | L_0.m = zm\}$ for any $z \in \mathbb{C}$ and define its *support* as the set $\mathrm{Supp}\, M = \{z \in \mathbb{C} | M_z \neq 0\}$.

Let $\mathcal{S}$ be the class of all $\mathbf{W}$-modules M such that

(i) $M = \oplus_{z\in\mathbb{C}} M_z$,

(ii) $\mathrm{Supp}\, M$ is exactly one $\mathbb{Z}$-coset, and

(ii) $\dim M_z = 1$ for all $z \in \mathrm{Supp}\,(M)$.

Here are three families of modules of the class $\mathcal{S}$:

1. The family of *tensor density modules* Ω_u^δ, where (δ, u) runs over $\mathbb{C}\times\mathbb{C}/\mathbb{Z}$. Here Ω_u^δ is the $\mathbf{W}$-module with basis $(e_z^\delta)_{z\in u}$ and action given by the formula:
$$L_m.e_z^\delta = (m\delta + z)e_{z+m}^\delta.$$
2. The *A-family* $(A_{a,b})_{(a,b)\in\mathbb{C}^2}$. Here $A_{a,b}$ is the $\mathbf{W}$-module with basis $(e_n^A)_{n\in\mathbb{Z}}$ and action given by the formula:
$$L_m.e_n^A = \begin{cases} (m+n)e_{m+n}^A & n \neq 0, \\ (am^2 + bm)e_m^A & n = 0. \end{cases}$$

3. The *B-family* $(B_{a,b})_{(a,b)\in\mathbb{C}^2}$. Here $B_{a,b}$ is the $\mathbf{W}$-module with basis $(e_n^B)_{n\in\mathbb{Z}}$ and action given by the formula:

$$L_m.e_n^B = \begin{cases} ne_{m+n}^B & n+m \neq 0, \\ (am^2+bm)e_0^B & n+m=0. \end{cases}$$

Set $\overline{A} := A/\mathbb{C}$. There are two exact sequences:

$$0 \longrightarrow \overline{A} \longrightarrow A_{a,b} \longrightarrow \mathbb{C} \longrightarrow 0, \text{ and}$$
$$0 \longrightarrow \mathbb{C} \longrightarrow B_{a,b} \longrightarrow \overline{A} \longrightarrow 0,$$

and we denote by $\operatorname{Res} : A_{a,b} \to \mathbb{C}$ the map defined by $\operatorname{Res} e_0^A = 1$ and $\operatorname{Res} e_n^A = 0$ if $n \neq 0$. These exact sequences do not split, except for $(a,b) = (0,0)$. Therefore the A-family is a deformation of $\Omega_0^1 \simeq A_{0,1}$ and the B-family is a deformation of of $\Omega_0^0 \simeq B_{0,1}$. Except the previous two isomorphisms and the obvious $A_{0,0} \cong B_{0,0} \cong \overline{A} \oplus \mathbb{C}$, there are some repetitions in the previous list due to the following isomorphisms:

1. the de Rham differential $d : \Omega_u^0 \to \Omega_u^1$, if $u \not\equiv 0 \bmod \mathbb{Z}$,

2. $A_{\lambda a,\lambda b} \cong A_{a,b}$ and $B_{\lambda a,\lambda b} \cong B_{a,b}$ for $\lambda \in \mathbb{C}^*$,

There are no other isomorphism in the class $\mathcal{S}$ beside those previously indicated. From now on, we will consider the couples $(a,b) \neq (0,0)$ as a projective coordinate, and the indecomposable modules in the AB-famillies are now parametrized by $\mathbb{P}^1$. Set $\infty = (0,1)$ and $\mathbb{A}^1 = \mathbb{P}^1 \setminus \infty$. Therefore the indecomposable $\mathbf{W}$-modules in the previous list, which are not tensor density modules, are the two $\mathbb{A}^1$-parametrized families $(A_\xi)_{\xi\in\mathbb{A}^1}$ and $(B_\xi)_{\xi\in\mathbb{A}^1}$, as in [MP]'s paper.

The classification of the $\mathbf{W}$-modules of the class $\mathcal{S}$ was given by I. Kaplansky and L. J. Santharoubane [KS], [K] (with a minor correction in [MP] concerning the parametrization of the AB-families):

Kaplansky-Santharoubane Theorem. *Let M be a $\mathbf{W}$-module of the class $\mathcal{S}$.*

1. *If M is irreducible, then there exists $(u,\delta) \in \mathbb{C}/\mathbb{Z} \times \mathbb{C}$, with $(u,\delta) \neq (0,0)$ or $(0,1)$ such that $M \simeq \Omega_u^\delta$.*

2. *If M is reducible and indecomposable, then M is isomorphic to either A_ξ or B_ξ for some $\xi \in \mathbb{P}^1$.*

3. *Otherwise, M is isomorphic to $\overline{A} \oplus \mathbb{C}$.*

1.2 Degree of the modules in the class $\mathcal{S}$

It follows from the previous remark that one can define *the degree* $\deg M$ for any $M \in \mathcal{S}$ as follows:

1. $\deg M = \delta$ if $M \cong \Omega_s^\delta$ for some $\delta \in \mathbb{C} \setminus \{0, 1\}$, and

2. $\deg M = \{0, 1\}$ otherwise.

By definition, the degree is a multivalued function. We also define *a degree* of M as a value $\delta \in \deg M$. Let $\mathcal{S}^*$ be the class of all pairs (M, δ), where $M \in \mathcal{S}$ and $\delta \in \deg M$. A pair $(M, \delta) \in \mathcal{S}^*$ will be often simplified as M and set $\deg M := \delta$. So, the degree function is a single valued function on $\mathcal{S}^*$. Usually, we consider Ω_s^δ as the element $(\Omega_s^\delta, \delta)$ of $\mathcal{S}^*$ for any δ.

For $M \in \mathcal{S}$, let M^* be its restricted dual, namely $M^* = \oplus_{x \in \mathbb{C}} M_x^*$. By definition, the class $\mathcal{S}$ is stable by the restricted duality and we have:

Lemma 1. *$(\Omega_u^\delta)^* \cong \Omega_{-u}^{1-\delta}$ and $(A_\xi)^* \cong B_\xi$.*

In particular, it follows that $\deg M^* = 1 - \deg M$ for any $M \in \mathcal{S}$.

2 Germs and bilinear maps

2.1 On the terminology '$\mathfrak{g}$-equivariant'

Throughout the whole paper, we will use the following convention. Let $\mathfrak{g}$ be a Lie algebra and let E be a $\mathfrak{g}$-module. When E is a space of maps, a $\mathfrak{g}$-invariant element of E will be called $\mathfrak{g}$-*equivariant*. We will use the same convention for spaces of germs of maps (see the definition below).

When $\mathfrak{g}$ is the one-dimensional Lie algebra $\mathbb{C}.L$, we will use the terminology *L-invariant* and *L-equivariant* instead of $\mathfrak{g}$-invariant and $\mathfrak{g}$-equivariant.

2.2 Weight modules and the $\mathfrak{S}_3$-symmetry

A $\mathbb{C}$-*graded vector space* is a vector space M endowed with a decomposition $M = \oplus_{z \in \mathbb{C}} M_z$ such that $\dim M_z < \infty$ for all $z \in \mathbb{C}$. Denote by $\mathcal{H}$ the category of all $\mathbb{C}$-graded vector spaces. It is convenient to denote by L_0 the degree operator, which acts as z on M_z. Given $M, N \in \mathcal{H}$, we denote by $\mathrm{Hom}_{L_0}(M, N)$ the space of L_0-equivariant linear maps from $\phi : M \to N$. Equivalently $\mathrm{Hom}_{L_0}(M, N)$ is the space of maps in the category $\mathcal{H}$.

By definition, a *Lie L_0-algebra* is a pair $(\mathfrak{g}, L_0)$, where $\mathfrak{g} = \oplus_{z\in\mathbb{C}}\, \mathfrak{g}_z$ is a $\mathbb{C}$-graded Lie algebra, L_0 is an element of $\mathfrak{g}_0$ such that $\mathrm{ad}(L_0)$ acts as the degree operator. A *weight* $\mathfrak{g}$-module is a $\mathbb{C}$-graded vector space M endowed with a structure of $\mathfrak{g}$-module. Of course it is required that L_0 acts as the degree operator on M and therefore we have $\mathfrak{g}_y.M_z \subset M_{y+z}$ for all y, $z \in \mathbb{C}$. Let $\mathcal{H}_{\mathfrak{g}}$ be the category of weight $\mathfrak{g}$-modules. For M and N in $\mathcal{H}_{\mathfrak{g}}$, denote by $\mathrm{Hom}_{\mathfrak{g}}(M, N)$ the space of $\mathfrak{g}$-equivariant linear maps from M to N.

Given M, N and P in $\mathcal{H}$, denote by $\mathbf{B}(M \times N, P)$ the space of all bilinear maps $\pi : M \times N \to P$ and by $\mathbf{B}_{L_0}(M \times N, P)$ the subspace of L_0-equivariant bilinear maps. Similarly if M, N and P are in $\mathcal{H}_{\mathfrak{g}}$, denote by $\mathbf{B}_{\mathfrak{g}}(M \times N, P)$ the space of $\mathfrak{g}$-equivariant bilinear maps.

For $P \in \mathcal{H}$, denote by P^* its restricted dual. By definition, we have $P^* = \oplus_{z\in\mathbb{C}}\, P^*_z$, where $P^*_z = (P_{-z})^*$.

Lemma 2. *Let M, N and P in $\mathcal{H}_{\mathfrak{g}}$. We have:*

$$\mathbf{B}_{\mathfrak{g}}(M \times N, P^*) \simeq \mathbf{B}_{\mathfrak{g}}(M \times P, N^*).$$

Proof. The lemma follows easily from the fact that

$$\mathbf{B}_{L_0}(M \times N, P^*) = \prod_{u+v+w=0} M^*_u \otimes N^*_v \otimes P^*_w.$$

□

It follows that $\mathbf{B}_{\mathfrak{g}}(M \times N, P^*)$ is fully symmetric in M, N and P. This fact will be referred to as the $\mathfrak{S}_3$*-symmetry.* The obvious symmetry $\mathbf{B}_{\mathfrak{g}}(M \times N, P) \simeq \mathbf{B}_{\mathfrak{g}}(N \times M, P)$ will be called the $\mathfrak{S}_2$*-symmetry*

2.3 Definition of germs

For $M \in \mathcal{H}$ and $x \in \mathbb{R}$, set $M_{\geq x} = \oplus_{\Re z \geq x}\, M_z$ and $M_{\leq x} = \oplus_{\Re z \leq x}\, M_z$, where $\Re z$ denotes the real part of z. Given another object $N \in \mathcal{H}$, let $\mathrm{Hom}^0(M, N)$ be the space of all linear maps $\phi : M \to N$ such that $\phi(M_{\geq x}) = 0$ for some $x \in \mathbf{R}$. Set $\mathcal{G}(M, N) = \mathrm{Hom}(M, N)/\mathrm{Hom}^0(M, N)$. The image in $\mathcal{G}(M, N)$ of some $\phi \in \mathrm{Hom}(M, N)$, which is denoted by $\mathcal{G}(\phi)$, is called its *germ.* The space $\mathcal{G}(M, N)$ is called the *space of germs of maps* from M to N.

Let $\mathfrak{g}$ be a Lie L_0-algebra and let M, $N \in \mathcal{H}_{\mathfrak{g}}$. It is clear that $\mathrm{Hom}^0(M, N)$ is a $\mathfrak{g}$-submodule of $\mathrm{Hom}(M, N)$ and thus $\mathfrak{g}$ acts on $\mathcal{G}(M, N)$. Denote by $\mathcal{G}_{\mathfrak{g}}(M, N)$ the space of $\mathfrak{g}$-equivariant germs. We will often use the following obvious fact: any $\psi \in \mathcal{G}_{L_0}(M, N)$ is the germ of a L_0-equivariant map $\phi : M \to N$, but in general a $\mathfrak{g}$-equivariant germ ψ is not the germ of a $\mathfrak{g}$-equivariant map ϕ.

Let M, $N \in \mathcal{H}$. A linear map $\phi : M \to N$ is called *continous* if for any $x \in \mathbb{R}$ there exist $y \in \mathbb{R}$ such that $\phi(M_{\geq y}) \subset N_{\geq x}$. The germ of a continous map ϕ is called a *continous germ* of a map.

It is not possible to compose arbitrary germs of maps. However let ϕ, ψ two morphisms of $\mathcal{H}$ such that the composition $\psi \circ \phi$ is defined. It is easy to show that $\mathcal{G}(\psi \circ \phi)$ only depends of $\mathcal{G}(\phi)$ and $\mathcal{G}(\psi)$ whenever ϕ is continous. Thus, it is possible to compose the continous germs.

Since the L_0-equivariant germs of maps are continous, the $\mathfrak{g}$-equivariant germs can be composed. Therefore we can define the category $\mathcal{G}(\mathcal{H}_{\mathfrak{g}})$ of *germs of weight* $\mathfrak{g}$*-modules* as follows. Its objects are weight $\mathfrak{g}$-modules, and for M, $N \in \mathcal{H}_{\mathfrak{g}}$, the space of $\mathcal{G}(\mathcal{H}_{\mathfrak{g}})$-morphisms from M to N is $\mathcal{G}_{\mathfrak{g}}(M, N)$. Viewed as an object of the category $\mathcal{G}(\mathcal{H}_{\mathfrak{g}})$, an object $M \in \mathcal{H}_{\mathfrak{g}}$ is called a *germ of a weight* $\mathfrak{g}$*-module* and it is denoted by $\mathcal{G}_{\mathfrak{g}}(M)$.

When $\mathfrak{g}_{\geq 0}$ and $\mathfrak{g}_{\leq 0}$ are finitely generated as Lie algebras, there is a concrete characterization of the $\mathfrak{g}$-equivariant germs of maps. Indeed let $M, N \in \mathcal{H}_{\mathfrak{g}}$ and let $\phi : M \to N$ be a L_0-equivariant map. Then $\mathcal{G}(\phi)$ is $\mathfrak{g}$-equivariant iff:

(i) the restriction $\phi : M_{\geq x} \to N_{\geq x}$ is $\mathfrak{g}_{\geq 0}$-equivariant, and

(ii) the induced map $\phi : M/M_{\leq x} \to N/N_{\leq x}$ is $\mathfrak{g}_{\leq 0}$-equivariant,

for any $x >> 0$.

2.4 Germs of modules of the class $\mathcal{S}$

It is easy to compute the germs of the $\mathbf{W}$-modules of the class $\mathcal{S}$.

Lemma 3. *For any $\xi_1, \xi_2 \in \mathbb{P}^1$, we have $\mathcal{G}_{\mathbf{W}}(A_{\xi_1}) = \mathcal{G}_{\mathbf{W}}(B_{\xi_2})$. Thus for any $M \in \mathcal{S}$, we have $\mathcal{G}_{\mathbf{W}}(M) \simeq \mathcal{G}_{\mathbf{W}}(\Omega_u^\delta)$ for some $\delta \in \mathbb{C}$ and $u \in \mathbb{C}/\mathbb{Z}$.*

The proof of the lemma follows easily from the definition. Recall that $\mathfrak{sl}(2)$ is the Lie subalgebra $\mathbb{C}L_{-1} \oplus \mathbb{C}L_0 \oplus \mathbb{C}L_1$ of $\mathbf{W}$. In what follows, it is useful to compare $\mathfrak{sl}(2)$-germs and $\mathbf{W}$-germs.

Lemma 4. *Let $\delta \in \mathbb{C}$ and $u \in \mathbb{C}/\mathbb{Z}$.*

(i) We have $\mathcal{G}_{\mathfrak{sl}(2)}(\Omega_u^\delta) \simeq \mathcal{G}_{\mathfrak{sl}(2)}(\Omega_u^{1-\delta})$.

(ii) If $\mathcal{G}_{\mathbf{W}_{\geq -1}}(\Omega_u^\delta) \simeq \mathcal{G}_{\mathbf{W}_{\geq -1}}(\Omega_u^\gamma)$ for some $\delta \neq \gamma$, then $\{\delta, \gamma\} = \{0, 1\}$.

Proof. Proof of the first assertion:

Choose a function $f : \mathbb{C} \to \mathbb{C}^*$ such that $f(z) = zf(z-1)$ whenever $\Re z > 1$. Define $\psi : \Omega_u^\delta \to \Omega_u^{1-\delta}$ by the formula:

$$\psi(e_z^\delta) = f(z-\delta)/f(z+\delta-1)\, e_z^{1-\delta},$$

for any $z \in u$. It is easy to check that $L_{\pm 1}.\psi(e_z^\delta) = \psi(L_{\pm 1}.e_z^\delta)$ whenever $z \in u$ and $\Re(z \pm \delta) > 1$. Therefore the germs of $L_{\pm 1}.\psi$ are zero, which means that $\mathcal{G}(\psi)$ is a $\mathfrak{sl}(2)$-equivariant isomorphism.

Proof of the second assertion: Assume that the $\mathbf{W}_{\geq -1}$-germs of Ω_u^δ and Ω_u^γ are isomorphic. Then there exist a L_0-equivariant isomorphism $\psi : \Omega_u^\delta \to \Omega_u^\gamma$ whose germ is $\mathbf{W}_{\geq -1}$-equivariant. It follows that

$\psi(L_1^2.e_z^\delta) = L_1^2.\psi(e_z^\delta)$ and $\psi(L_2.e_z^\delta) = L_2.\psi(e_z^\delta)$,

for any $z \in u$ with $\Re z >> 0$. Set $\psi(e_z^\delta) = ae_z^\gamma$ and $\psi(e_{z+2}^\delta) = be_z^\gamma$. It follows that:

$(z+\delta)(z+1+\delta)b = (z+\gamma)(z+1+\gamma)a$, and

$(z+2\delta)b = (z+2\gamma)a$.

Therefore we get:

$(z+\delta)(z+1+\delta)(z+2\gamma) = (z+\gamma)(z+1+\gamma)(z+2\delta)$.

Since this identity holds for any $z \in u$ with $\Re z >> 0$, it is valid for any z. Since $\delta \neq \gamma$, it follows easily that $\{\delta, \gamma\} = \{0, 1\}$. $\square$

It follows from Lemma 4(ii) that the degree of modules of the class $\mathcal{S}$ is indeed an invariant of their $\mathbf{W}$-germs.

2.5 Germs of bilinear maps

For M, N and $P \in \mathcal{H}$, denote by $\mathbf{B}(M \times N, P)$ the space of bilinear maps from $M \times N$ to P. Also denote by $\mathbf{B}^0(M \times N, P)$ the space of all $\pi \in \mathbf{B}(M \times N, P)$ such that $\pi(M_{\geq x} \times N_{\geq x}) = 0$ for any $x >> 0$. Set

$$\mathcal{G}(M \times N, P) = \mathbf{B}(M \times N, P)/\mathbf{B}^0(M \times N, P),$$

The image of a bilinear map $\pi \in \mathbf{B}(M \times N, P)$ in $\mathcal{G}(M \times N, P)$ is called its *germ* and it is denoted by $\mathcal{G}(\pi)$. The set $\mathcal{G}(M \times N, P)$ is called the *space of germs of bilinear maps* from $M \times N$ to P.

Let $\mathfrak{g}$ be a Lie L_0-algebra and let M, N and P be weight $\mathfrak{g}$-modules. As before, $\mathcal{G}(M \times N, P)$ is a $\mathfrak{g}$-module in a natural way and we denote by $\mathcal{G}_\mathfrak{g}(M \times N, P)$ the space of $\mathfrak{g}$-equivariant germs of bilinear maps. As before,the composition of $\mathfrak{g}$-equivariant germs of bilinear maps with $\mathfrak{g}$-equivariant germs of linear maps is well defined. Thus we obtain:

Lemma 5. *The space $\mathcal{G}_\mathfrak{g}(M \times N, P)$ depends functorially on the germs of the weight $\mathfrak{g}$-modules M, N and P.*

Let M, N, P, and Q in $\mathcal{S}$ and let $\phi \in \mathrm{Hom}_{L_0}(P, Q)$. Assume that $\mathcal{G}(\phi)$ is a $\mathfrak{sl}(2)$-equivariant isomorphism. The composition with ϕ induces a map

$$\mathcal{G}(\phi)_* : \mathcal{G}_{\mathfrak{sl}(2)}(M \times N, P) \to \mathcal{G}_{\mathfrak{sl}(2)}(M \times N, Q).$$

Lemma 6. *Assume that $\mathcal{G}(\phi)$ is not $\mathbf{W}_{\geq -1}$-equivariant. Then the two subspaces $\mathcal{G}_{\mathbf{W}}(M \times N, Q)$ and $\mathcal{G}(\phi)_* \mathcal{G}_{\mathbf{W}}(M \times N, P)$ of $\mathcal{G}_{\mathfrak{sl}(2)}(M \times N, Q)$ have a zero intersection.*

Proof. The lemma is equivalent to the following statement: for any L_0-equivariant bilinear map $\pi : M \times N \to P$ whose germ is $\mathbf{W}$-equivariant and non-zero, $\mathcal{G}(\phi \circ \pi)$ is not $\mathbf{W}$-equivariant. So, we prove this statement.

Set $\mu = L_2.(\phi \circ \pi)$. We claim that $\mathcal{G}(\mu) \neq 0$, i.e. for any $r \in \mathbb{R}$ there are scalars x, y with $\Re x > r$ and $\Re y > r$ such that $\mu(M_x \times N_y) \neq 0$ Indeed, if r is big enough we have

(i) the restriction $\pi_{\geq r} : M_{\geq r} \times N_{\geq r} \to P_{\geq 2r}$ of π is $\mathbf{W}_{\geq 0}$-equivariant, and

(ii)$L_1.P_z = P_{z+1}$ for any z with $\Re z > 2r$.

Since $\mathcal{G}(\pi) \neq 0$, there exists $(x_0, y_0) \in \mathrm{Supp}\, M \times \mathrm{Supp}\, N$ with $\Re x_0 > r$, $\Re y_0 > r$ and $\pi(M_{x_0} \times N_{y_0}) = P_{z_0}$, where $z_0 = x_0 + y_0$.

By hypothesis $\mathcal{G}(\phi)$ is $\mathfrak{sl}(2)$-equivariant but not $\mathbf{W}_{\geq -1}$-equivariant. Since $\mathbf{W}_{\geq -1}$ is generated by $\mathfrak{sl}(2)$ and L_2, we have $\mathcal{G}(L_2.\phi) \neq 0$. Hence there exists $k \in \mathbf{Z}_{\geq 0}$ such that $(L_2.\phi)(P_{k+z_0}) \neq 0$.

By assumptions, the linear span of $\cup_{m.n \geq 0}\, \pi(M_{x_0+m} \times N_{y_0+n})$ is a $\mathbf{W}_{\geq 0}$-module and the $\mathbf{W}_{\geq 0}$-module $P_{\geq \Re z_0}$ is generated by $P_{z_0} = \pi(M_{x_0} \times N_{y_0})$. Thus there are $m, n \in \mathbb{Z}_{\geq 0}$ with $m + n = k$ such that $\pi(M_{x_0+m} \times N_{y_0+n}) = P_{z_0+k}$.

Since $L_2.(\phi \circ \pi_{\geq r}) = (L_2.\phi) \circ \pi_{\geq r}$, it follows that $\mu(M_{x_0+m} \times N_{y_0+n}) \neq 0$, which proves the claim. $\square$

3 Degenerate and non-degenerate bilinear maps

In this section, we define the notions of *degenerate* and *non-degenerate* bilinear maps and similar notions for germs of bilinear maps. We show that a $\mathbf{W}$-equivariant bilinear map π between modules of the class $\mathcal{S}$ is degenerate if and only if $\mathcal{G}(\pi) = 0$. Moreover, $\mathcal{G}(\pi)$ is non-degenerate if $\mathcal{G}(\pi) \neq 0$.

Let M, N and P in $\mathcal{H}$. For $\pi \in \mathbf{B}(M \times N, P)$, the set $\mathrm{Supp}\, \pi = \{(x, y) |\, \pi(M_x \times N_y) \neq 0\}$ is called the *support* of π. The bilinear map π is called *non-degenerate* if $\mathrm{Supp}\, \pi$ is Zarisky dense in $\mathbb{C}^2$. Otherwise, it is called *degenerate*.

Any germ $\tau \in \mathcal{G}(M \times N, P)$ is represented by a bilinear map $\pi \in \mathbf{B}(M \times N, P)$, and let $\pi_{\geq x}$ be its restriction to $M_{\geq x} \times N_{\geq x}$. The germ τ is called *non-degenerate* if $\pi_{\geq x}$ is non-degenerate for any $x >> 0$.

From now on, assume that M, N and P are $\mathbf{W}$-modules of the class $\mathcal{S}$. For $\pi \in \mathbf{B}_{\mathbf{W}}(M \times N, P)$, set $M_\pi = \{m \in M | \, \pi(m \times N_{\geq x}) = 0 \, \text{for} \, x >> 0\}$ and $N_\pi = \{n \in N | \, \pi(M_{\geq x} \times n) = 0 \, \text{for} \, x >> 0\}$. It is clear that M_π and N_π are $\mathbf{W}$-submodules.

Lemma 7. *Let $\pi \in \mathbf{B}^0_{\mathbf{W}}(M \times N, P)$. Then we have:*

(i) $\pi(M_\pi \times N_\pi) \subset P^{\mathbf{W}}$ and therefore $\pi(M_\pi \times N_\pi)$ is isomorphic to 0 or $\mathbb{C}$.

(ii) M/M_π is isomorphic to 0 or $\mathbb{C}$.

(iii) N/N_π is isomorphic to 0 or $\mathbb{C}$.

Proof. It follows from the explicit description of all modules $X \in \mathcal{S}$ (see Section 1) that

1. if Y is a $\mathbf{W}$-submodule with $L_0.Y \neq 0$, then X/Y is isomorphic to $\mathbb{C}$ or 0, and
2. if $x \in X$ satisfies $L_k.x = 0$ for $k >> 0$, then x is $\mathbf{W}$-invariant.

Since $\mathcal{G}(\pi)$ is zero, M_π contains $M_{\geq x}$ for any $x >> 0$. Therefore, $L_0.M_\pi \neq 0$ and Assertions (ii) and (iii) follows.

Moreover for any $(m, n) \in M_\pi \times N_\pi$, we have $L_k.\pi(m, n) = \pi(L_k.m, n) + \pi(m, L_k.n) = 0$ for $k >> 0$. Thus $\pi(m, n)$ is $\mathbf{W}$-invariant which proves the first assertion. □

Let $\pi \in \mathbf{B}_{\mathbf{W}}(M \times N, P)$ with $\mathrm{Supp}\, \pi \subset \{(0, 0)\}$. Obviously there are $\mathbf{W}$-equivariant maps $a : M \to \mathbb{C}$, $b : N \to \mathbb{C}$ and $c : \mathbb{C} \to P$ such that $\pi(l, m) = c(a(l)b(m))$. Since it comes from a bilinear map between trivial modules, such a bilinear map π will be called *trivial*. Note that non-zero trivial maps only occur when M and N are in the A-family and P is in the B-family.

For a subset Z of $\mathbb{C}^2$, denotes by $\overline{Z}$ its Zariski closure. Also define the three lines H, V and D of $\mathbb{C}^2$ by:

$$H = \mathbb{C} \times \{0\}, \; V = \{0\} \times \mathbb{C} \text{ and } D = \{(z, -z) | \, z \in \mathbb{C}\}.$$

Lemma 8. *Let $\pi \in \mathbf{B}^0_{\mathbf{W}}(M \times N, P)$ and set $S = \overline{\mathrm{Supp}\ \pi}$. Assume that π is not trivial. Then we have:*

(i) S is a union of lines and $\mathrm{Supp}\ \pi \subset H \cup D \cup V$.

(ii) $\pi(M_\pi \times N_\pi) \neq 0$ iff $D \subset S$.

(iii) $M/M_\pi \neq 0$ iff $V \subset S$.

(iv) $N/N_\pi \neq 0$ iff $H \subset S$.

Proof. By Lemma 7, we have $\pi(M_\pi \times N_\pi) = 0$ or $\mathbb{C}$. Note that π induces the two bilinear maps $\eta : M_\pi \times N_\pi \to \pi(M_\pi \times N_\pi)$ and $\theta : M \times N \to P/\pi(M_\pi \times N_\pi)$.

Step 1: We claim that $\eta = 0$ or the bilinear map η has infinite rank. Assume otherwise. By Lemma 7, the image of η is $\mathbb{C}$. Since η factors through finite dimensional modules, it follows that M_π has a finite dimensional quotient. By Lemma 7 (ii), M_π is infinite dimensional. Hence M_π is reducible, which implies that $M = M_\pi$. Similarly, $N = N_\pi$. It follows easily that π is a trivial bilinear map which contradicts the hypothesis.

It follows that $\eta = 0$ or $\overline{\mathrm{Supp}\ \eta} = D$.

Step 2: We claim that $\overline{\mathrm{Supp}\ \pi} = \overline{\mathrm{Supp}\ \theta} \cup \overline{\mathrm{Supp}\ \eta}$.

Since $\pi(M_\pi \times N_\pi) \simeq \mathbb{C}$ or 0, we have: $\overline{\mathrm{Supp}\ \theta} \cup \overline{\mathrm{Supp}\ \eta} \subset \overline{\mathrm{Supp}\ \pi} \subset \overline{\mathrm{Supp}\ \theta} \cup D$. Therefore the claim follows from the previous step.

Step 3: Using the short exact sequence

$$\begin{aligned} 0 \longrightarrow & M \otimes N/M_\pi \otimes N_\pi \longrightarrow M \otimes (N/N_\pi) \oplus (M/M_\pi) \otimes N \longrightarrow \\ & \longrightarrow M/M_\pi \otimes N/N_\pi \longrightarrow 0, \end{aligned}$$

it follows that, up to the point $(0,0)$, the sets $\mathrm{Supp}\ \theta$ and $\mathrm{Supp}\ M \times \mathrm{Supp}\ (N/N_\pi) \cup \mathrm{Supp}\ (M/M_\pi) \times \mathrm{Supp}\ N$ coincide, which proves the lemma. ☐

Lemma 9. *A bilinear map $\pi \in \mathbf{B}_{\mathbf{W}}(M \times N, P)$ is degenerate iff its germ is zero. Moreover, if $\mathcal{G}(\pi) \neq 0$, the germ $\mathcal{G}(\pi)$ is non-degenerate.*

Proof. Let $\pi \in \mathbf{B}_{\mathbf{W}}(M \times N, P)$. For $x \in \mathbb{R}$, denote by $\pi_{\geq x}$ the restriction of π to $M_{\geq x} \times N_{\geq x}$. It is enough to prove the second assertion.

First Step: We claim that if (s,t) belongs to $\mathrm{Supp}\ \pi_{\geq 1}$, then $(s,t) + H$ or $(s,t) + V$ lies in $\overline{\mathrm{Supp}\ \pi_{\geq 1}}$. By hypothesis, $\Re(s+t) > 0$ and it follows from the explicit description of modules of the class $\mathcal{S}$ that $L_k.P_{s+t} \neq 0$ for any $k >> 0$. So we get

$\pi(L_k.M_s \times N_t) \neq 0$ or $\pi(M_s \times L_k.N_t) \neq 0$ for $k >> 0$.

Hence $(s+k, t)$ or $(s, t+k)$ is in Supp π_x for infinitely many $k > 0$, and the claim follows.

Second step: Assume that $\mathcal{G}(\pi)$ is non-zero and prove that its germ is non-degenerate. Let $x \geq 1$ be an arbitrary real number, and let $(s, t) \in$ Supp $M_{\geq x} \times$ Supp $N_{\geq x}$. By definition there exists two increasing sequences of integers $0 \leq a_1 < a_2 \ldots$ and $0 \leq b_1 < b_2 \ldots$ such that $(s+a_k, t+b_k)$ belongs to Supp $\pi_{\geq x}$ for all k. Since all lines $(s+a_k, t+b_k)+V$, $(s+a_k, t+b_k)+H$ are distinct, $\overline{\text{Supp } \pi_{\geq x}}$ contains infinitely many lines, and therefore $\overline{\text{Supp } \pi_{\geq x}} = \mathbb{C}^2$ □

4 Examples of W-equivariant bilinear maps

This section provides a list of **W**-equivariant bilinear maps between modules of the class $\mathcal{S}$. The goal of this paper is to prove that this list generates all bilinear maps between modules of the class $\mathcal{S}$. More precisely, if one allows the following operations: the $\mathfrak{S}_3$-symmetry, the composition with morphisms between modules in $\mathcal{S}$ and the linear combination, then one obtains all bilinear maps between modules of the class $\mathcal{S}$.

4.1 The Poisson algebra $\mathcal{P}$ of symbols twisted pseudo-differential operators

To be brief, we will not give the definition of the algebra $\mathcal{D}$ of twisted pseudo-differential operators on the circle, see e.g [IM]. Just say that the term *twisted* refers to the fact that complex powers of z and $\frac{d}{dz}$ are allowed. As usual, $\mathcal{D}$ is an associative filtered algebra whose associated graded space $\mathcal{P}$ is a Poisson algebra.

Indeed $\mathcal{P}$ is explicitely defined as follows. As vector space, $\mathcal{P}$ has basis the family $(z^s \partial^\delta)$ where s and δ runs over $\mathbb{C}$ (here ∂ stands for the symbol of $\frac{d}{dz}$). The commutative associative product on $\mathcal{P}$ is denoted by . and the Lie bracket is denoted by $\{,\}$. These products are explicitely defined on the basis elements by

$$(z^s \partial^\delta).(z^{s'} \partial^{\delta'}) = (z^{s+s'} \partial^{\delta+\delta'}),$$
$$\{(z^s \partial^\delta), (z^{s'} \partial^{\delta'})\} = (\delta s' - \delta' s)\, z^{s+s'-1} \partial^{\delta+\delta'-1}.$$

It is clear that $\oplus_{n\in\mathbb{Z}}\,\mathbb{C}\,z^{n+1}\partial$ is a Lie subalgebra naturally isomorphic to $\mathbf{W}$. As a $\mathbf{W}$-module, there is a decomposition of $\mathcal{P}$ as

$$\mathcal{P} = \oplus_{(\delta,u)\in\mathbb{C}\times\mathbb{C}/\mathbb{Z}}\ \Omega_u^\delta,$$

where $\Omega_u^\delta = \oplus_{s\in u}\,\mathbb{C}\,z^{s-\delta}\partial^{-\delta}$. We have

$$\Omega_u^{\delta_1}.\Omega_v^{\delta_2} \subset \Omega_{u+v}^{\delta_1+\delta_2},$$
$$\{\Omega_u^{\delta_1}, \Omega_v^{\delta_2}\} \subset \Omega_{u+v}^{\delta_1+\delta_2+1}$$

for all $\delta_1, \delta_2 \in \mathbb{C}$ and $u, v \in \mathbb{C}/\mathbb{Z}$. Therefore the Poisson structure induces two families of $\mathbf{W}$-equivariant bilinear maps:

$$P_{u,v}^{\delta_1,\delta_2} : \Omega_u^{\delta_1} \times \Omega_v^{\delta_2} \to \Omega_{u+v}^{\delta_1+\delta_2}, (m,n) \mapsto m.n,$$
$$B_{u,v}^{\delta_1,\delta_2} : \Omega_u^{\delta_1} \times \Omega_v^{\delta_2} \to \Omega_{u+v}^{\delta_1+\delta_2+1}, (m,n) \mapsto \{m,n\}.$$

It is clear that all these maps are non-degenerate, except $B_{u,v}^{0,0}$. Indeed we have $B_{u,v}^{0,0} = 0$, for all $u, v \in \mathbb{C}/\mathbb{Z}$.

4.2 The extended Lie algebra $\mathcal{P}_\xi$

Recall Kac's construction of an extended Lie algebra [Kac]. Start with a triple $(\mathfrak{g}, \kappa, \delta)$, where $\mathfrak{g}$ is a Lie algebra, $\delta : \mathfrak{g} \to \mathfrak{g}, x \mapsto \delta.x$ is a derivation and $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a symmetric $\mathfrak{g}$-equivariant and δ-equivariant bilinear form. Then the extended Lie algebra is the vector space $\mathfrak{g}_e = \mathfrak{g} \oplus \mathbb{C}\,\delta \oplus \mathbb{C}\,c$ and its Lie bracket $[,]_e$ is defined by the following relations:

$[x,y]_e = [x,y] + \kappa(x, \delta.y)c,$
$[\delta, x]_e = \delta.x,$
$[c, \mathfrak{g}_e]_e = 0,$

for any $x, y \in \mathfrak{g}$ and where $[x,y]$ is the Lie bracket in $\mathfrak{g}$.

We will apply this construction to the Lie algebra $\mathcal{P}$. The residue map $\mathrm{Res} : \mathcal{P} \to \mathbb{C}$ is defined as follows: $\mathrm{Res}(\Omega_u^\delta) = 0$ for $(\delta, u) \neq (1, 0)$ and the restriction of Res to Ω_0^1 is the usual residue. Thus set $\kappa(x,y) = \mathrm{Res}\, xy$ for any x, $y \in \mathcal{P}$. Let (a,b) be projective coordinates of $\xi \in \mathbb{P}^1$. Define the derivation δ_ξ of $\mathcal{P}$ by $\delta_\xi x = z^{b-a}\partial^{-a}\{z^{a-b}\partial^a, x\}$ for any $x \in \mathcal{P}$. Informally, we have $\delta_\xi x = \{\log(z^{a-b}\partial^a), x\}$.

It is easy to check that κ is equivariant under $\mathrm{ad}(\mathcal{P})$ and under δ_ξ. The two-cocycle $x, y \in \mathcal{P} \mapsto \mathrm{Res}(x\delta_\xi y)$ is the Khesin-Kravchenko cocycle [KK]. The corresponding extended Lie algebra will be denoted by $\mathcal{P}_\xi$. Thus we have

$$\mathcal{P}_\xi = \mathcal{P} \oplus \mathbb{C}\,\delta_\xi \oplus \mathbb{C}\,c.$$

Since $\mathcal{P}_\xi$ is not a Poisson algebra, its Lie bracket will be denoted by $[\,,]$. Set $\mathcal{P}_\xi^+ = [\mathcal{P}_\xi, \mathcal{P}_\xi]$ and $\mathcal{P}_\xi^- = \mathcal{P}_\xi/Z(\mathcal{P}_\xi)$, where $Z(\mathcal{P}_\xi)$ is the center of $\mathcal{P}_\xi$. As before $\mathbf{W}$ is a Lie subalgebra of $\mathcal{P}_\xi$, and the $\mathbf{W}$-modules $\mathcal{P}_\xi^\pm$ decomposes as follows

$$\mathcal{P}_\xi^\pm = \oplus_{(\delta,u)\in\mathbb{C}\times\mathbb{C}/\mathbb{Z}}\ \Omega_u^\delta(\xi,\pm),$$

where $\Omega_u^\delta(\xi,\pm)=\Omega_u^\delta$ or all $(\delta,u)\in\mathbb{C}\times\mathbb{C}/\mathbb{Z}$ except that $\Omega_0^0(\xi,-)\simeq A_\xi$ and $\Omega_0^1(\xi,+)\simeq B_\xi$. The Lie bracket of $\mathcal{P}_\xi$ induces a bilinear map

$$B(\xi):\mathcal{P}_\xi^-\times\mathcal{P}_\xi^-\to\mathcal{P}_\xi^+, (m \,\text{modulo}\, Z(\mathcal{P}_\xi), n \,\text{modulo}\, Z(\mathcal{P}_\xi)\mapsto[m,n].$$

As before, the components of $B(\xi)$ provide the following $\mathbf{W}$-equivariant bilinear maps

$$B_{u,v}^{\delta_1,\delta_2}(\xi):\Omega_u^{\delta_1}(\xi,-)\times\Omega_v^{\delta_2}(\xi,-)\to\Omega_{u+v}^{\delta_1+\delta_2+1}(\xi,+), (m,n)\mapsto[m,n].$$

It should be noted that if $\delta_1\delta_2(\delta_1+\delta_2)\neq 0$, then we have $B_{u,v}^{\delta_1,\delta_2}(\xi)=B_{u,v}^{\delta_1,\delta_2}$.

4.3 Other W-equivariant bilinear maps.

The Grozman operator: Among the $\mathbf{W}$-equivariant bilinear maps between modules of the class $\mathcal{S}$, the most surprizing is the Grozman operator. It is the bilinear map $G_{u,v}:\Omega_u^{-2/3}\times\Omega_v^{-2/3}\to\Omega_{u+v}^{5/3}$, defined by the following formula

$$G_{u,v}(e_x^{-2/3},e_y^{-2/3})=(x-y)(2x+y)(x+2y)e_{x+y}^{5/3}.$$

The bilinear map Θ_∞: Let $\xi\in\mathbb{P}^1$ with projective coordinates (a,b). Define $\Theta_\xi: A_{a,b}\times A_{a,b}\to B_{a,b}$ by the following requirements:

$\Theta_\xi(u_m^A,u_n^A)=0$ if $mn(m+n)\neq 0$ or if $m=n=0$

$\Theta_\xi(u_0^A,u_m^A)=-\Theta(u_m^A,u_0^A)=1/m\,u_m^B$ if $m\neq 0$,

$\Theta_\xi(u_{-m}^A,u_m^A)=1/m\,u_0^B$ if $m\neq 0$.

It is easy to see that $a\Theta_\xi$ is identical to the bracket $-B_{0,0}^{0,0}(a,b)$. In particular, Θ_ξ is $\mathbf{W}$-equivariant (for $a=0$, this follows by extension of polynomial identities). So Θ_∞ is the only new bilinear map, since, for $\xi\neq\infty$, Θ_ξ is essentially the bracket of $\mathcal{P}_\xi$.

The bilinear map $\eta(\xi_1,\xi_2,\xi_3)$: Let ξ_1,ξ_2,ξ_3 be points in $\mathbb{P}^1$ which are not all equal, with projective coordinates (a_1,b_1), (a_2,b_2), (a_3,b_3). Choose a non-zero triple (x,y,z) such that $z(a_3,b_3)=x(a_1,b_1)+y(a_2,b_2)$. Recall that all A_ξ have the same underlying vector space, therefore the map

$$y\mathrm{Res}\times id + x id\times\mathrm{Res}: A_{a_1,b_1}\times A_{a_2,b_2}\to A_{a_3,b_3}$$

is well defined. This defines a map, up to a scalar multiple,

$$\eta(\xi_1,\xi_2,\xi_3):A_{\xi_1}\times A_{\xi_2}\to A_{\xi_3},$$

which is clearly $\mathbf{W}$-equivariant.

The obvious map P^M: Also for each $M\in\mathcal{S}$ denote by P^M the obvious map $((a+x),m)\in(\overline{A}\oplus\mathbb{C})\times M\mapsto xm\in M$.

4.4 Primitive bilinear maps

Let $\mathcal{N}$ be the class of all non-zero $\mathbf{W}$-equivariant maps $\phi : M \to N$, where M, N are non-isomorphic modules of the class $\mathcal{S}$. Up to conjugacy, there are only two possibilities:

(i) M is in the A-family, N is in the B-family and ϕ is the morphism $M \twoheadrightarrow \mathbb{C} \hookrightarrow N$, or

(ii) M is in the B-family, N is in the A-family and ϕ is the morphism $M \twoheadrightarrow \overline{A} \hookrightarrow N$.

Hence, the condition $M \not\simeq N$ means that M and N are not simultaneously isomorphic to $\overline{A} \oplus \mathbb{C}$.

Let M, M', N, N', P and P' in $\mathcal{S}$. Let $\phi : M' \to M$, $\psi : N' \to N$ and $\theta : P \to P'$ be $\mathbf{W}$-equivariant maps, and let $\pi \in \mathbf{B}_{\mathbf{W}}(M \times N, P)$ be a $\mathbf{W}$-equivariant bilinear map. Set $\pi' = \theta \circ \pi \circ (\phi \times \psi)$. If at least one of the three morphisms ϕ, ψ or θ is of the class $\mathcal{N}$, then π' is called an *imprimitive form* of π.

A $\mathbf{W}$-equivariant bilinear map between modules of the class $\mathcal{S}$ is called *primitive* if is not a linear combinations of imprimitive forms. The composition of any three composable morphisms of the class $\mathcal{N}$ is zero. It follows easily that any $\mathbf{W}$-equivariant bilinear maps between modules of the class $\mathcal{S}$ is either primitive, or it is a linear combination of imprimitive bilinear form. Thus the classification of all $\mathbf{W}$-equivariant bilinear maps between modules of the class $\mathcal{S}$ reduces to the classification of primitive ones.

Lemma 10. *Let M, N and P be $\mathbf{W}$ modules of the class $\mathcal{S}$, and let $\pi \in \mathbf{B}_{\mathbf{W}}(M \times N, P)$. Assume one of the following conditions holds*

(i) M and N are irreducible and P is the linear span of $\pi(M \times N)$

(ii) N and P are irreducible and the left kernel of π is zero.

Then π is primitive.

This obvious lemma is useful to check easily that some bilinear maps are primitive. Some of the bilinear forms defined in this section are not primitive. It will be proved in Corollary 1 of Section 11 that, up to $\mathfrak{S}_3$-symmetry, all primitive bilinear forms between modules of the class $\mathcal{S}$ have been defined in this section.

4.5 Examples of W-equivariant germs

In what follows, the elements of $\mathbb{C}^3$ will be written as triples $(\delta_1, \delta_2, \gamma)$. Let σ be the involution defined by $(\delta_1, \delta_2, \gamma)^\sigma = (\delta_2, \delta_1, \gamma)$. Let $\mathfrak{z}$ be the union of the two affine planes H_i, the six affine lines D_i and the five points P_i defined as follows. For $i = 0, 1$, the plane H_i is defined by the equation $\gamma = \delta_1 + \delta_2 + i$. The six lines D_i are parametrized as follows:

$D_1 = \{(0, \delta, \delta + 2) | \delta \in \mathbb{C}\}$ and $D_2 = D_1^\sigma$,
$D_3 = \{(\delta, 1, \delta) | \delta \in \mathbb{C}\}$ and $D_4 = D_3^\sigma$,
$D_5 = \{(\delta, -(1 + \delta), 1) | \delta \in \mathbb{C}\}$
$D_6 = \{(\delta, 1 - \delta, 0) | \delta \in \mathbb{C}\}$.

The fives points are $P_1 = (0, 0, 3)$, $P_2 = (0, -2, 1)$, $P_3 = P_2^\sigma$ and $P_4 = (1, 1, 0)$ and $P_5 = (-2/3, -2/3, 5/3)$. Also let $\mathfrak{z}^*$ be the set of all $(\delta_1, \delta_2, \gamma) \in \mathfrak{z}$ such that $\{\delta_1, \delta_2, \gamma\} \not\subset \{0, 1\}$.

In what follows, we will consider Ω_u^δ as a module of the class $\mathcal{S}^*$. Thus we can define without ambiguity the degree of any $\pi \in \mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^\gamma)$ as the scalar $\gamma - \delta_1 - \delta_2$. The following table provide a list of germs π in $\mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^\gamma)$, when $(\delta_1, \delta_2, \gamma)$ runs over $\mathfrak{z}^*$. In the table, we omit the symbol $\mathcal{G}$. For example, d^{-1} stands for $\mathcal{G}(d)^{-1}$, which is well-defined even for $u \equiv 0 \operatorname{mod} \mathbb{Z}$.

Table 1: List of $\pi \in \mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^\gamma)$, where $(\delta_1, \delta_2, \gamma)$ runs over $\mathfrak{z}^*$

	$\deg \pi$	$(\delta_1, \delta_2, \gamma)$	π
1.	3	$(-\frac{2}{3}, -\frac{2}{3}, \frac{5}{3})$	$G_{u,v}$
2.	3	$(0, 0, 3)$	$B_{u,v}^{1,1} \circ (d \times d)$
3.	3	$(0, -2, 1)$	$d \circ B_{u,v}^{1,-2} \circ (d \times id)$
4.	3	$(-2, 0, 1)$	$d \circ B_{u,v}^{-2,1} \circ (id \times d)$
5.	2	$(0, \delta, \delta + 2)$	$B_{u,v}^{1,\delta} \circ (d \times id)$
6.	2	$(\delta, 0, \delta + 2)$	$B_{u,v}^{\delta,1} \circ (id \times d)$
7.	2	$(\delta, -\delta - 1, 1)$	$d \circ B_{u,v}^{\delta,-\delta-1}$
8.	1	$(\delta_1, \delta_2, \delta_1 + \delta_2 + 1)$	$B_{u,v}^{\delta_1,\delta_2}$
9.	0	$(\delta_1, \delta_2, \delta_1 + \delta_2)$	$P_{u,v}^{\delta_1,\delta_2}$
10.	-1	$(1, \delta, \delta)$	$P_{u,v}^{0,\delta} \circ (d^{-1} \times id)$
11.	-1	$(\delta, 1, \delta)$	$P_{u,v}^{\delta,0} \circ (id \times d^{-1})$
12.	-1	$(\delta, 1 - \delta, 0)$	$d^{-1} \circ P_{u,v}^{\delta,1-\delta}$

The condition $(\delta_1, \delta_2, \gamma) \in \mathfrak{z}^*$ implies that $(\delta_1, \delta_2) \neq (0, 0)$ in the line 8, $(\delta_1, \delta_2) \neq (0, 0), (0, 1)$ or $(1, 0)$ in the line 9, $\delta \neq 0$ or 1 in the lines 10-12.

Let M, N and P be $\mathbf{W}$-modules of the class $\mathcal{S}$. Set $u = \operatorname{Supp} M$, $v =$

Supp N and assume that Supp $P = u + v$. Let $\delta_1 \in \deg M$, $\delta_2 \in \deg N$ and $\gamma \in \deg P$. It follows from Lemma 3 that

$$\mathcal{G}(M) = \mathcal{G}(\Omega_u^{\delta_1}),\ \mathcal{G}(N) = \mathcal{G}(\Omega_v^{\delta_2}) \text{ and } \mathcal{G}(P) = \mathcal{G}(\Omega_{u+v}^{\gamma}).$$

Lemma 11. *Assume that* $(\delta_1, \delta_2, \gamma) \in \mathfrak{z}$. *Then* $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ *is not zero, and moreover we have* $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) \geq 2$ *if* $\{\delta_1, \delta_2, \gamma\} \subset \{0, 1\}$. *More precisely we have:*

(i) for $(\delta_1, \delta_2, \gamma) \in \mathfrak{z}^*$, *Table 1 provides a non-zero* $\pi \in \mathcal{G}_{\mathbf{W}}(M \times N, P)$,

(ii) if $\{\delta_1, \delta_2, \gamma\} \subset \{0, 1\}$, *then we have* $\mathcal{G}(M) = \mathcal{G}(\Omega_u^0)$, $\mathcal{G}(N) = \mathcal{G}(\Omega_v^0)$ *and* $\mathcal{G}(P) = \mathcal{G}(\Omega_{u+v}^1)$ *and the maps* $\pi_1 := P_{u,v}^{0,1} \circ (id \times d)$ *and* $\pi_2 := P_{u,v}^{1,0} \circ (d \times id)$ *are non-proportional elements of* $\mathcal{G}_{\mathbf{W}}(M \times N, P)$.

Theorem 2, proved in Section 7, states that the maps listed in the previous lemma provide a basis of $\mathcal{G}_{\mathbf{W}}(M \times N, P)$. It also states that $\mathcal{G}_{\mathbf{W}}(M \times N, P) = 0$ if $(\delta_1, \delta_2, \gamma) \notin \mathfrak{z}$.

5 Classification of W-equivariant degenerate bilinear maps

Let M, N and P be in the class $\mathcal{S}$. The goal of the section is the classification of all **W**-equivariant degenerate bilinear maps $\pi : M \times N \to P$. In order to simplify the statements, we will always assume that M, N and P are indecomposable.

Assume that $\pi \neq 0$ and set $S = \overline{\text{Supp } \pi}$. By Lemma 8, $S \subset H \cup D \cup V$ is an union of lines. Thus there are four cases, of increasing complexity:

(i) $S = 0$,

(ii) S consists of one line H, D or V,

(iii) S consists of two lines among H, D and V, or

(iv) $S = H \cup D \cup V$.

Since the three lines H, V and D are exchanged by the $\mathfrak{S}_3$-symmetry, we can reduce the full classification to the following four cases:

(i) $S = 0$,

(ii) $S = V$,

(iii) $S = H \cup V$,

(iv) $S = H \cup V \cup D$.

In the following lemmas, it is assumed that $\pi : M \times N \to P$ is a non-zero degenerate $\mathbf{W}$-equivariant bilinear map.

Lemma 12. *If $S = 0$, then π is trivial.*

The lemma is obvious.

A $\mathbf{W}$-equivariant map $\psi : N \to P$ is called an *almost-isomorphism* if its kernel has dimension ≤ 1. The only almost-isomorphisms which are not isomorphisms between modules of the class $\mathcal{S}$ are the maps from B_ξ to A_η obtained as the composition of $B_\xi \twoheadrightarrow \overline{A}$ and $\overline{A} \hookrightarrow A_\eta$.

Lemma 13. *Assume that $S = V$. Then there is surjective maps $\phi : M \to \mathbb{C}$ and almost-isomorphism $\psi : N \to P$ such that*

$$\pi(x, y) = \phi(x)\psi(y),$$

for any $(x, y) \in M \times N$.

Lemma 13 is obvious.

In order to investigate the case where S contains two lines, it is necessary to state a diagram chasing lemma. Let $\mathfrak{g}$ be a Lie algebra, let X be a $\mathfrak{g}$-module. For $\xi \in H^1(\mathfrak{g}, M)$, denote by X_ξ the corresponding extension:

$$0 \to X \to X_\xi \to \mathbb{C} \to 0.$$

Lemma 14. *Assume that $End_{\mathfrak{g}}(X) = \mathbb{C}$ and that $X = \mathfrak{g}.X$. Let $\xi_1, \xi_2, \xi_3 \in H^1(\mathfrak{g}, X)$ and set $Z = X_{\xi_1} \otimes X_{\xi_2} / X \otimes X$. We have*

$$\dim \mathrm{Hom}_{\mathfrak{g}}(Z, X_{\xi_3}) = 3 - r,$$
$$\dim \mathrm{Hom}_{\mathfrak{g}}(Z, X) = 2 - s,$$

where r is the rank of $\{\xi_1, \xi_2, \xi_3\}$ and s is the rank of $\{\xi_1, \xi_2\}$ in $H^1(\mathfrak{g}, M)$.

Proof. For $i = 1$ to 3, there are elements $\delta_i \in X_{\xi_i} \setminus X$ such that the map $x \in \mathfrak{g} \to x.\delta_i \in X$ is a cocycle of the class ξ_i. Since $X_{\xi_i} = X \oplus \mathbb{C}\delta_i$, we have

$$Z = [\delta_1 \otimes X] \oplus [X \otimes \delta_2] \oplus \mathbb{C}(\delta_1 \otimes \delta_2).$$

Let $\pi \in \mathrm{Hom}_{\mathfrak{g}}(Z, X_{\xi_3})$. Note that $\delta_1 \otimes X$ is a submodule of Z isomorphic with X. Since $\mathfrak{g}.X = X$, we have $\pi(\delta_1 \otimes X) \subset X$. Thus there exists $\lambda \in \mathbb{C}$ such that $\pi(\delta_1 \otimes x) = \lambda x$ for all $x \in X$. Similarly, there exists $\mu \in \mathbb{C}$ such that $\pi(x \otimes \delta_2) = \mu x$ for all $x \in X$. By definition, we have $\pi(\delta_1 \otimes \delta_2) = \nu\delta_3 + x_0$ for some $\nu \in \mathbb{C}$ and some x_0 in X.

The $\mathfrak{g}$-equivariance of π is equivalent to the equation:

$$\mu g.\delta_1 + \lambda g.\delta_2 = \nu g.\delta_3 + g.x_0 \text{ for all } g \in \mathfrak{g}.$$

Thus $\dim \mathrm{Hom}_{\mathfrak{g}}(Z, X_{\xi_3})$ is exactly the dimension of the space of triples $(\lambda, \mu, \nu) \in \mathbb{C}^3$ such that

$$\lambda\delta_2 + \mu\delta_1 - \nu\delta_3 \equiv 0 \text{ in } H^1(\mathfrak{g}, X),$$

and the first assertion follows. The second assertion is similar. □

For $\xi \in \mathbb{P}^1$ and $t \in \mathbb{C}$, define $\eta^t_{(\xi)} : A_\xi \times A_\xi \to A_\xi$ by the formula

$$\eta^t_\xi(m, n) = \mathrm{Res}(m)n + t\mathrm{Res}(n)m,$$

and recall that $\eta(\xi_1, \xi_2, \xi_3)$ is defined in Section 4.3.

Lemma 15. *Assume that $S = V \cup H$. Then π is conjugate to one of the following:*

(i) $\eta(\xi_1, \xi_2, \xi_3)$, for some $\xi_1, \xi_2, \xi_3 \in \mathbb{P}^1$ with $\xi_3 \notin \{\xi_1, \xi_2\}$, or

(ii) η^t_ξ for some $t \neq 0$ and $\xi \in \mathbb{P}^1$.

Proof. By Lemma 8 we have $\pi(M_\pi \times N_\pi) = 0$, $M/M_\pi = \mathbb{C}$ and $N/N_\pi = \mathbb{C}$. It follows that $M \simeq A_{\xi_1}$ and $N \simeq A_{\xi_2}$ for some $\xi_1, \xi_2 \in \mathbb{P}^1$. Thus $(M/M_\pi) \otimes N_\pi$ is isomorphic to $\overline{A}$. Since π induces a non-zero map $(M/M_\pi) \otimes N_\pi \to P$, the $\mathbf{W}$-module P contains $\overline{A}$, hence P is isomorphic to A_{ξ_3} for some non-zero $\xi_3 \in \mathbb{P}^1$.

Let $B = \{\mu \in \mathbf{B_W}(M \times N, P) | \mu(M_\pi \times N_\pi) = 0\}$. It follows from the Kaplansky-Santharoubane Theorem that $\dim H^1(W, \overline{A}) = 2$. Thus if ξ_1, ξ_2, ξ_3 are not all equal, it follows from Lemma 14 that $B = \mathbb{C}\,\eta(\xi_1, \xi_2, \xi_3)$. However Supp $\eta(\xi_1, \xi_2, \xi_3)$ lies inside H or V if $\xi_1 = \xi_3$ or $\xi_2 = \xi_3$. Hence we have $\xi_3 \notin \{\xi_1, \xi_2\}$ and π is conjugate to $\eta(\xi_1, \xi_2, \xi_3)$. Similarly, if all ξ_i are equal to some $\xi \in \mathbb{P}^1$, then B is the two dimensional vector space generated by the affine line $\{\eta^t_\xi | t \in \mathbb{C}\}$. Thus π is conjugate to some η^t_ξ. Moreover the hypothesis Supp $\pi = H \cup V$ implies that $t \neq 0$. □

Lemma 16. *Assume that $S = V \cup H \cup D$. Then, π is conjugate to Θ_ξ for some $\xi \in \mathbb{P}^1$, modulo a trivial map.*

Proof. It follows from Lemma 8 that we have $\pi(M_\pi \times N_\pi) \cong \mathbb{C}$, $M/M_\pi \cong \mathbb{C}$ and $N/N_\pi \cong \mathbb{C}$. Thus $M = A_{\xi_1}$, $N = A_{\xi_2}$ and $P = B_{\xi_3}$ form some $\xi_i \in \mathbb{P}^1$.

Set $Z = A_\xi \otimes A_\xi / \overline{A} \otimes \overline{A}$. By Lemma 8, the composition of any $\mu \in \mathbf{B}^0_{\mathbf{W}}(A_{\xi_1} \times A_{\xi_2}, B_{\xi_3})$ with the map $B_{\xi_3} \to \overline{A}$ provides a linear map $\overline{\mu} : Z \to \overline{A}$.

Since $\overline{\mu}$ is not zero, Lemma 14 implies that $\xi_1 = \xi_2$. By the $\mathfrak{S}_3$-symmetry, B_{ξ_3} is the restricted dual of A_{ξ_2}, so we have $\xi_3 = \xi_1$. Set $\xi := \xi_1 = \xi_2 = \xi_3$ and consider the following exact sequence

$$0 \to \mathbf{B}_{\mathbf{W}}(A_\xi \times A_\xi, \mathbb{C}) \to \mathbf{B}^0_{\mathbf{W}}(A_\xi \times A_\xi, B_\xi) \to \mathrm{Hom}_{\mathbf{W}}(Z, B_\xi/\mathbb{C}),$$

where the last arrow is the map $\mu \mapsto \overline{\mu}$.

It is clear that the subspace $\mathbf{B}_{\mathbf{W}}(A_\xi \times A_\xi, \mathbb{C})$ of $\mathbf{B}^0_{\mathbf{W}}(A_\xi \times A_\xi, B_\xi)$ is the space of trivial bilinear maps. By the previous lemma, $\mathrm{Hom}_{\mathbf{W}}(Z, B_\xi/\mathbb{C})$ has dimension one. It follows from its definition that $\mathrm{Supp}\,\Theta_\xi = H \cup V \cup D$. Hence we have

$$\mathbf{B}^0_{\mathbf{W}}(A_\xi \times A_\xi, B_\xi) = \mathbf{B}_{\mathbf{W}}(A_\xi \times A_\xi, \mathbb{C}) \oplus \mathbb{C}\Theta_\xi.$$

Hence π is conjugate to Θ_ξ modulo a trivial map. $\square$

Theorem 1. *Let M, N and P indecomposable modules of the class $\mathcal{S}$ and let $\pi : M \times N \to P$ be a $\mathbf{W}$-equivariant degenerate bilinear map. Up to the $\mathfrak{S}_3$-symmetry, π is conjugate to one of the following:*

(i) a trivial bilinear map $\pi : A_{\xi_1} \times A_{\xi_2} \to B_{\xi_3}$,

(ii) the map $\pi : A_\xi \times N \to P$, $(m, n) \mapsto \mathrm{Res}(m)\psi(n)$ where $\xi \in \mathbb{P}^1$ and where $\psi : N \to P$ is an almost-isomorphism,

(iii) the map $\eta(\xi_1, \xi_2, \xi_3)$ for some ξ_1, ξ_2 and $\xi_3 \in \mathbb{P}^1$ with $\xi_3 \notin \{\xi_1, \xi_2\}$, or the map η^t_ξ for some $\xi \in \mathbb{P}^1$ and $t \neq 0$,

(iv) $\Theta_\xi + \tau$, where $\xi \in \mathbb{P}^1$ and τ is a trivial map.

The following table is another presentation of Theorem 1. To limit the number of cases, the list is given up to the $\mathfrak{S}_3$-symmetry. That is why the datum in the third column is P^* and not P.

Table 2: **List, up to the $\mathfrak{S}_3$-symmetry, of possible S and d, where $d = \dim \mathbf{B}^0_{\mathbf{W}}(M \times N, P)$ and $S = \overline{\mathrm{Supp}\ \pi}$ for $\pi \in \mathbf{B}^0_{\mathbf{W}}(M \times N, P)$.**

	$M \times N$	P^*	S	d	Restrictions
1.	$A_{\xi_1} \times A_{\xi_2}$	A_{ξ_3}	0	1	$Card\{\xi_1, \xi_2, \xi_3\} \geq 2$
2.	$A_\xi \times X$	X^*	V	1	$X \not\simeq A_\xi$ or B_ξ
3.	$A_{\xi_1} \times B_{\xi_2}$	B_{ξ_3}	V	1	
4.	$A_{\xi_1} \times A_{\xi_2}$	B_{ξ_3}	$H \cup V$	1	$\xi_3 \notin \{\xi_1, \xi_2\}$
5.	$A_\xi \times A_\xi$	B_ξ	H, V or $H \cup V$	2	
6.	$A_\xi \times A_\xi$	A_ξ	0 or $H \cup V \cup D$	2	

Except the indicated restrictions, $X \in \mathcal{S}$ is arbitrary and ξ, ξ_1, ξ_2 and ξ_3 are arbitrary.

6 Bounds for the dimension of the spaces of germs of bilinear maps

Let M, N and P be $\mathbf{W}$-modules of the class $\mathcal{S}$. In this section we introduce a space $\tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(M \times N, P)$ which is a good approximation of $\mathcal{G}_{\mathbf{W}}(M \times N, P)$. Indeed we have:

$$\mathcal{G}_{\mathbf{W}}(M \times N\, P) \subset \tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(M \times N\, P) \subset \mathcal{G}_{\mathfrak{sl}(2)}(M \times N, P).$$

6.1 On $\dim \mathcal{G}_{\mathfrak{sl}(2)}(M \times N, P)$

For an element $\gamma \in \mathbb{C}^2$, set $L.\gamma = \gamma + (1,0)$ and $R.\gamma = \gamma + (0,1)$. Similarly, for a pair $\{\alpha, \beta\}$ of elements of $\mathbb{C}^2$, set $L.\{\alpha, \beta\} = \{L.\alpha, L.\beta\}$ and $R.\{\alpha, \beta\} = \{R.\alpha, R.\beta\}$. The pair $\{\alpha, \beta\}$ is called *adjacent* if $\beta = \alpha + (1, -1)$ or $\alpha = \beta + (1, -1)$. Thus $L.\{\alpha, \beta\}$ and $R.\{\alpha, \beta\}$ are adjacent whenever $\{\alpha, \beta\}$ is adjacent. Given $(x, y) \in \mathbb{C}^2$, let $C(x, y)$ be the set of all elements of $\mathbb{C}^2$ of the form $(x + m, y + n)$ with $m, n \in \mathbb{Z}_{\geq 0}$ and $(m, n) \neq (0, 0)$.

Let δ_1, δ_2 and γ be scalar, and let u and v be $\mathbb{Z}$-cosets. Let $\pi : \Omega_u^{\delta_1} \times \Omega_v^{\delta_2} \to \Omega_{u+v}^{\gamma}$ be an L_0-equivariant bilinear map. Let $(e_x^{\delta_1})_{x \in u}$ be the basis of $\Omega_u^{\delta_1}$ defined in Section 1. Similarly denote by $(e_y^{\delta_2})_{y \in v}$ and $(e_z^{\gamma})_{z \in u+v}$ the corresponding bases of $\Omega_v^{\delta_2}$ and Ω_{u+v}^{γ}.

Since π is L_0-equivariant, there exists a function $X : u \times v \to \mathbb{C}$ defined by the identity:

$$\pi(e_x^{\delta_1}, e_y^{\delta_2}) = X(x, y) e_{x+y}^{\gamma}.$$

Lemma 17. *Assume there exists $(x, y) \in u \times v$ such that:*
(i) $\operatorname{Supp}(L_{\pm 1}.\pi) \cap C(x, y) = \emptyset$,
(ii) $\Re x > \pm \Re \delta_1$, $\Re y > \pm \Re \delta_2$, $\Re(x + y) > \pm \Re \gamma$, *and*
(iii) $X(x + 1, y) = X(x, y + 1) = 0$.
Then $\mathcal{G}(\pi) = 0$

Proof. First step: We claim that for any adjacent pair $\{\alpha, \beta\}$ in $C(x, y)$ with $X(\alpha) = X(\beta) = 0$, then X vanishes on $R^k.L^l.\{\alpha, \beta\}$ for any $k, l \in \mathbb{Z}_{\geq 0}$.

First prove that X vanishes on $L.\{\alpha, \beta\}$ Indeed we can assume that $\alpha = \beta + (1, -1)$, and therefore we have $\alpha = (x' + 1, y')$, $\beta = (x', y' + 1)$ for some $(x', y') \in \mathbb{C}^2$.

Since $L_{-1}.\pi(e_{x'+1}^{\delta_1}, e_{y'+1}^{\delta_2})$ is a linear combination of $\pi(e_{x'}^{\delta_1}, e_{y'+1}^{\delta_2})$ and $\pi(e_{x'+1}^{\delta_1}, e_{y'}^{\delta_2})$, we get

$$0 = L_{-1}.\pi(e_{x'+1}^{\delta_1}, e_{y'+1}^{\delta_2}) = X(x' + 1, y' + 1) L_{-1}.e_{x'+y'+2}^{\gamma}.$$

Since $\Re(x'+y'+2-\gamma)>0$, we get $L_{-1}.e^{\gamma}_{x'+y'+2}\neq 0$ and therefore $X(x'+1,y'+1)=0$. Moreover we have

$$0=L_1.\pi(e^{\delta_1}_{x'+1},e^{\delta_2}_{y'})=\pi(L_1.e^{\delta_1}_{x'+1},e^{\delta_2}_{y'})+\pi(e^{\delta_1}_{x'+1},L_1.e^{\delta_2}_{y'}).$$

Using that $X(x'+1,y'+1)=0$, it follows that $\pi(L_1.e^{\delta_1}_{x'+1},e^{\delta_2}_{y'})=0$. Since $\Re(x'+1+\delta_1)>0$, we have $L_1.e^{\delta_1}_{x'+1}\neq 0$ and therefore $X(x'+2,y')=0$. Hence X vanishes on $L.\{\alpha,\beta\}$.

Similarly, X vanishes on $R.\{\alpha,\beta\}$. It follows by induction that X vanishes on $R^k.L^l.\{\alpha,\beta\}$ for any $k,l\in\mathbb{Z}_{\geq 0}$.

Second step: Set $\alpha=(x+1,y)$ and $\beta=(x,y+1)$. We have $C(x,y)=\cup_{k,l\in\mathbb{Z}_{\geq 0}} R^k.L^l.\{\alpha,\beta\}$. It follows that X vanishes on $C(x,y)$. Hence $\mathcal{G}(\pi)=0$. □

Let $\pi\in\mathbf{B}_{L_0}(\Omega^{\delta_1}_u\times\Omega^{\delta_2}_v,\Omega^{\gamma}_{u+v})$. As before, set

$$\pi(e^{\delta_1}_x,e^{\delta_2}_y)=X(x,y)e^{\gamma}_{x+y}.$$

Lemma 18. *Assume that $\mathcal{G}(\pi)$ is $\mathfrak{sl}(2)$-equivariant and non-zero. Then there exists $(x_0,y_0)\in u\times v$ such that*

$$X(\alpha)\neq 0 \text{ or } X(\beta)\neq 0,$$

for any adjacent pair $\{\alpha,\beta\}$ in $C(x_0,y_0)$.

Proof. Since $\mathcal{G}(\pi)$ is $\mathfrak{sl}(2)$-equivariant, we can choose $(x_0,y_0)\in u\times v$ such that:

(i) $\operatorname{Supp} L_{\pm 1}\circ\pi\cap C(x_0,y_0)=\emptyset$

Moreover we can assume that $\Re x_0$ and $\Re y_0$ are big enough in order that:

(ii) $\Re x_0>\pm\Re\delta_1$, $\Re y_0>\pm\Re\delta_2$, $\Re(x_0+y_0)>\pm\Re\gamma$.

Let $\{(x+1,y),(x,y+1)\}$ be an adjacent pair in $C(x_0,y_0)$. The couple (x,y) satisfies the conditions (i) and (ii) of the previous lemma. Since $\mathcal{G}(\pi)\neq 0$, it follows that $X(x+1,y)\neq 0$ or $X(x,y+1)\neq 0$. □

Let M,N and P be $\mathbf{W}$-modules of the class $\mathcal{S}$.

Lemma 19. *We have* $\dim\mathcal{G}_{\mathfrak{sl}(2)}(M\times N,P)\leq 2$.

Proof. Let π_1,π_2,π_3 be any elements in $\mathbf{B}_{L_0}(M\times N,P)$ such that $\mathcal{G}(\pi_i)\in\mathcal{G}_{\mathfrak{sl}(2)}(M\times N,P)$.

Since the space $\mathcal{G}(M\times N,P)$ only depends on the germs of M, N and P, we can assume that $M=\Omega^{\delta_1}_u$, $N=\Omega^{\delta_2}_v$ and $P=\Omega^{\gamma}_w$ for some scalars δ_1,δ_2 and γ and some $\mathbb{Z}$-cosets u, v and w. Moreover, we can assume $w=u+v$, since otherwise it is obvious that $\mathcal{G}_{\mathfrak{sl}(2)}(M\times N,P)=0$.

There is $(x, y) \in u \times v$ such that $\mathrm{Supp}\,(L_{\pm 1}.\pi_i) \cap C(x, y) = \emptyset$ for $i = 1$ to 3. Adding some positive integers to x and y if necessary, we can assume that $\Re x > \pm\Re\delta_1$, $\Re y > \pm\Re\delta_2$ and $\Re(x + y) > \pm\Re\gamma$. There is a non-zero triple (a, b, c) of scalars with:

$$[a\pi_1 + b\pi_2 + c\pi_3](e^{\delta_1}_{x+1}, e^{\delta_2}_{y}) = [a\pi_1 + b\pi_2 + c\pi_3](e^{\delta_1}_{x}, e^{\delta_2}_{y+1}) = 0.$$

It follows from Lemma 17 that $a\mathcal{G}(\pi_1)+b\mathcal{G}(\pi_2)+c\mathcal{G}(\pi_3) = 0$. Since any three arbitrary elements $\mathcal{G}(\pi_1), \mathcal{G}(\pi_2)$ and $\mathcal{G}(\pi_3)$ of $\mathcal{G}_{\mathfrak{sl}(2)}(M \times N, P)$ are linearly dependant, it follows that $\dim \mathcal{G}_{\mathfrak{sl}(2)}(M \times N, P) \leq 2$.

$\square$

6.2 The recurrence relations

Let M, N and P be three $\mathbf{W}$-modules of the class $\mathcal{S}$. Let $\pi \in \mathbf{B}(M \times N, P)$. Set

$$\tilde{\pi}(m, n) = L_{-2}L_2.\pi(m, n) - \pi(L_{-2}L_2.m, n) - \\ -\pi(m, L_{-2}L_2.n) - \pi(L_{-2}.m, L_2.n) - \pi(L_2.m, L_{-2}.n),$$

for all $(m, n) \in M \times N$. Note that we have

$$\tilde{\pi} = L_{-2} \circ (L_2.\pi) + (L_{-2}.\pi) \circ (L_2 \times \mathrm{id}) + (L_{-2}.\pi) \circ (\mathrm{id} \times L_2).$$

Similarly, for a germ $\mu \in \mathcal{G}(M \times N, P)$, set

$$\tilde{\mu} = L_{-2} \circ (L_2.\mu) + (L_{-2}.\mu) \circ (L_2 \times \mathrm{id}) + (L_{-2}.\mu) \circ (\mathrm{id} \times L_2).$$

Set $\tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(M \times N, P) = \{\mu \in \mathcal{G}_{\mathfrak{sl}(2)}(M \times N, P) |\, \tilde{\mu} = 0\}$. It follows from the definitions that we have

$$\mathcal{G}_{\mathbf{W}}(M \times N, P) \subset \tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(M \times N, P) \subset \mathcal{G}_{\mathfrak{sl}(2)}(M \times N, P).$$

Let δ_1, δ_2 and γ be three scalars. For $k = 1$, 2, set

$$\begin{aligned} a_k(x, y) &= (x + k\delta_1)(y - k\delta_2), \\ b_k(x, y) &= k^2(\delta_1 + \delta_2 - \gamma - \delta_1^2 - \delta_2^2 + \gamma^2) - 2xy, \\ c_k(x, y) &= (x - k\delta_1)(y + k\delta_2). \end{aligned}$$

Let u and v be two $\mathbb{Z}$-cosets and let $\pi : \Omega^{\delta_1}_u \times \Omega^{\delta_2}_v \to \Omega^{\gamma}_{u+v}$ be a L_0-equivariant bilinear map. As before, define the function X by the identity:

$$\pi(e^{\delta_1}_x, e^{\delta_2}_y) = X(x, y)e^{\gamma}_{x+y}.$$

Lemma 20. *Assume that the $\mathcal{G}(\pi)$ belongs to $\tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(\Omega^{\delta_1}_u \times \Omega^{\delta_2}_v, \Omega^{\gamma}_{u+v})$. Then there exists $(x_0, y_0) \in u \times v$ such that:*

$$a_k(x, y)X(x + k, y - k) + b_k(x, y)X(x, y) + c_k(x, y)X(x - k, y + k) = 0,$$

for $k = 1, 2$ and all $(x, y) \in C(x_0, y_0)$.

Proof. For $k = 1, 2$, set $\pi_k = L_{-k} \circ (L_k.\pi) + (L_{-k}.\pi) \circ (L_k \times id) + (L_{-k}.\pi) \circ (id \times L_k)$. Since the germ of π is $\mathfrak{sl}(2)$-equivariant we have $\mathcal{G}(\pi_1) = 0$ and since $\pi_2 = \tilde{\pi}$ we also have $\mathcal{G}(\pi_2) = 0$. Therefore there exist $(x_0, y_0) \in u \times v$ such that $\operatorname{Supp} \pi_1$ and $\operatorname{Supp} \pi_2$ do not intersect $C(x_0, y_0)$. This condition is equivalent to the recurrence relations of the lemma. □

6.3 The case $\dim \tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(M \times N, P) = 2$

Lemma 21. *Let M, N and P be $\mathbf{W}$-modules of the class $\mathcal{S}$. If*

$$\dim \tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(M \times N, P) = 2,$$

then one of the following assertions holds:

(i) $\deg M = \deg N = \deg P = \{0, 1\}$,

(ii) $\deg M = -1/2$, $\deg N = \{0, 1\}$ *and* $\deg P \in \{-1/2, 3/2\}$

(iii) $\deg M = \{0, 1\}$, $\deg N = -1/2$, *and* $\deg P \in \{-1/2, 3/2\}$.

Proof. Assume that $\dim \tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(M \times N, P) = 2$. We can assume that $M = \Omega_u^{\delta_1}$, $N = \Omega_v^{\delta_2}$ and $P = \Omega_{u+v}^{\gamma}$ for some δ_1, δ_2 and γ in $\mathbb{C}$ and some $u, v \in \mathbb{C}/\mathbb{Z}$. Choose $\pi_1, \pi_2 \in \mathbf{B}_{L_0}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma})$ whose germs form a basis of $\tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma})$. For $i = 1, 2$ define the functions $X_i(x, y)$ by the identity

$$\pi_i(e_x^{\delta_1}, e_y^{\delta_2}) = X_i(x, y) e_{x+y}^{\gamma}.$$

By Lemma 20, there exists $(x_0, y_0) \in u \times v$ such that

(1) $\quad a_1(x, y)X_i(x+1, y-1) + b_1(x, y)X_i(x, y) + c_1(x, y)X_i(x-1, y+1) = 0,$

(2) $\quad a_2(x, y)X_i(x+2, y-2) + b_2(x, y)X_i(x, y) + c_2(x, y)X_i(x-2, y+2) = 0,$

for $i = 1, 2$ and all $(x, y) \in C(x_0, y_0)$. Moreover by Lemma 17, we can assume that the vectors $(X_1(x+1, y), X_1(x, y+1))$ and $(X_2(x+1, y), X_2(x, y+1))$ are linearly independant, for all $(x, y) \in C(x_0, y_0)$.

First step: We claim that

$$a_2(x, y)b_1(x+1, y-1)c_1(x-1, y+1)c_1(x, y) = \\ a_1(x, y)a_1(x+1, y+1)b_1(x-1, y+1)c_2(x, y),$$

for all $(x, y) \in \mathbb{C}^2$, where the functions a_i, b_i and c_i are defined in the previous section.

From now on, we assume that (x, y) belongs to $C(x_0 + 1, y_0 + 1)$. To simplify the expressions in the proof, we set $A_{il} = a_i(x + l, y - l)$, $B_{il} = b_i(x + l, y - l)$ and $C_{il} = c_i(x + l, y - l)$, for any $l \in \{-1, 0, 1\})$. Thus Identity (2) can be written as:

$$A_{2,0}X_i(x+2, y-2) + *X_i(x, y) + C_{2,0}X_i(x-2, y+2) = 0,$$

Here and in what follows, $*$ stands for a certain constant whose explicit value is not important at this stage. Multiplying by $A_{1,1}C_{1,-1}$, we obtain

(3) $A_{2,0}C_{1,-1}[A_{1,1}X_i(x+2,y-2)] + *X_i(x,y) +$
$$A_{1,1}C_{2,0}[C_{1,-1}X_i(x-2,y+2)] = 0,$$

Note that $(x+1,y-1)$ and $(x-1,y+1)$ belong to $C(x_0,y_0)$. Using Relation (1) we have

(4) $$A_{1,1}X_i(x+2,y-2) + B_{1,1}X_i(x+1,y-1) + *X_i(x,y) = 0$$

(5) $$*X_i(x,y) + B_{1,-1}X_i(x-1,y+1) + C_{1,-1}X_i(x-2,y+2) = 0$$

With Relations (4) and (5) we can eliminate the terms $[A_{1,1}X_i(x+2,y-1)]$ and $[C_{1,-1}X_i(x-2,y+2)]$ in Relation (3). We obtain

(6) $A_{2,0}B_{1,1}C_{1,-1}X_i(x+1,y-1) + *X_i(x,y) +$
$$A_{1,1}B_{1,-1}C_{2,0}X_i(x-1,y+1) = 0.$$

Moreover Relation (1) can be written as

(7) $$A_{1,0}X_i(x+1,y-1) + *X_i(x,y) + C_{1,0}X_i(x-1,y+1) = 0,$$

Thus Relations (6) and (7) provide two linear equations connecting $X_i(x+1,y-1)$, $X_i(x,y)$ and $X_i(x-1,y+1)$. Since $(x,y),(x+1,y-1)$ is an adjacent pair, it follows that the two triples $(X_1(x+1,y-1), X_1(x,y), X_1(x-1,y+1))$ and $(X_2(x+1,y-1), X_2(x,y), X_2(x-1,y+1))$ are linearly independent. So the linear relations (6) and (7) are proportional, which implies that
$$A_{2,0}B_{1,1}C_{1,-1}C_{1,0} = A_{1,0}A_{1,1}B_{1,-1}C_{2,0},$$
or equivalently

(8) $a_2(x,y)b_1(x+1,y-1)c_1(x-1,y+1)c_1(x,y) =$
$$a_1(x,y)a_1(x+1,y+1)b_1(x-1,y+1)c_2(x,y).$$

This identity holds for all $(x,y) \in C(x_0+1,y_0+1)$. Since $C(x_0+1,y_0+1)$ is Zariski dense in $\mathbb{C}^2$, Identity (8) holds for any $(x,y) \in \mathbb{C}^2$.

Second step: We claim that
$$\delta_1 + \delta_2 - \gamma - \delta_1^2 - \delta_2^2 + \gamma^2 = 0.$$

Assume otherwise and set $\tau = \delta_1 + \delta_2 - \gamma - \delta_1^2 - \delta_2^2 + \gamma^2$. We have $b_1(x,y) = \tau - 2xy$, therefore the polynomial b_1 is irreducible. Observe that all irreducible factors of the left side of (8) are degree 1 polynomials, except $b_1(x+1,y-1)$ and all irreducible factors of the right side side of (8) are degree 1 polynomials, except $b_1(x-1,y+1)$. Hence the irreducible factors of both sides do not coincide, which proves the claim.

Third step: We claim that δ_1 and δ_2 belong to $\{-1/2, 0, 1\}$. Using that $\delta_1 + \delta_2 - \gamma - \delta_1^2 - \delta_2^2 + \gamma^2 = 0$, Identity (8) looks like

$(x+2\delta_1)(x+1)(x-1-\delta_1)(x-\delta_1)f(y)$
$$= (x+\delta_1)(x+1+\delta_1)(x-1)(x-2\delta_1)g(y)$$

where $f(y)$ and $g(y)$ are some functions of y. Since $x+1$ is a factor of the left side of the identity, it follows that $\delta_1 = 1$, 0 or $-1/2$. The proof that $\delta_2 = 1$, 0 or $-1/2$ is identical.

Fourth step: We claim that the case $\delta_1 = \delta_2 = -1/2$ is impossible. The Equations (6) and (7) can be written as

(6) $$aX_i(x+1, y-1) + bX_i(x, y) + *X_i(x-1, y+1) = 0,$$

(7) $$cX_i(x+1, y-1) + dX_i(x, y) + *X_i(x-1, y+1) = 0,$$

where a, b, c and d are explicit functions of x and y (as before, * denotes some functions which are irrelevant for the present computation). Using that $\delta_1 + \delta_2 - \gamma - \delta_1^2 - \delta_2^2 + \gamma^2 = 0$ and a brute force computation, we obtain

$$ad - bc = 9/32(1+2y)(2x-1)(2xy-1).$$

So the Equations (6) and (7) are not proportional, which contradicts that $(X_1(x+1, y-1), X_1(x, y))$ and $(X_2(x+1, y-1), X_2(x, y))$ are independent.

Final step: If δ_1 and δ_2 belongs to $\{0, 1\}$, then $\gamma^2 - \gamma = 0$ i.e $\gamma = 0$ or 1 and the triple $(\delta_1, \delta_2, \gamma)$ satisfies Assertion (i). If $\delta_1 = -1/2$, then $\delta_2 = 0$ or 1 and $\gamma^2 - \gamma = 3/4$, i.e. $\gamma = -1/2$ or $3/2$ and the triple $(\delta_1, \delta_2, \gamma)$ satisfies Assertion (ii). Similarly if $\delta_2 = -1/2$, the triple $(\delta_1, \delta_2, \gamma)$ satisfies Assertion (iii). $\square$

6.4 The case $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) = 2$

Let $M, N, P \in \mathcal{S}$ with $\operatorname{Supp} P = \operatorname{Supp} M + \operatorname{Supp} N$.

Lemma 22. *We have* $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) = 2$ *iff* $\deg M = \deg N = \deg P = \{0, 1\}$.

Proof. Set $u = \operatorname{Supp} M$ and $v = \operatorname{Supp} N$. By Lemma 19, we have $\dim \mathcal{G}_{\mathfrak{sl}(2)}(M \times N, P) \leq 2$ and therefore $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) \leq 2$.

First step: Assume that $\deg M = \deg N = \deg P = \{0, 1\}$. By Lemma 11 we have $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) \geq 2$. Thus $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) = 2$.

Second step: Set $d^- = \dim \mathcal{G}_{\mathbf{W}}(\Omega_u^0 \times \Omega_v^{-1/2}, \Omega_{u+v}^{-1/2})$ and $d^+ = \dim \mathcal{G}_{\mathbf{W}}(\Omega_u^0 \times \Omega_v^{-1/2}, \Omega_{u+v}^{3/2})$. We claim that $d^+ = d^- = 1$.

By Lemma 4, there is a $\mathfrak{sl}(2)$-equivariant isomorphism $\phi : \mathcal{G}(\Omega^{-1/2}) \to \mathcal{G}(\Omega^{3/2})$. By Lemma 6, $\phi_* \mathcal{G}_{\mathbf{W}}(\Omega_u^0 \times \Omega_v^{-1/2}, \Omega_{u+v}^{-1/2})$ and $\mathcal{G}_{\mathbf{W}}(\Omega_u^0 \times \Omega_v^{-1/2}, \Omega_{u+v}^{3/2})$ are two subspaces of $\mathcal{G}_{\mathfrak{sl}(2)}(\Omega_u^0 \times \Omega_v^{-1/2}, \Omega_{u+v}^{3/2})$ with trivial intersection. Thus we have $d^+ + d^- \leq 2$. However by Lemma 11, we have $d^+ \geq 1$ and $d^- \geq 1$. It follows that $d^+ = d^- = 1$.

Third step: Conversely, assume that $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) = 2$. It follows that $\dim \tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(M \times N, P) = 2$.

By the previous step, the case (ii) of the assertion of Lemma 21 cannot occur. Using the $\mathfrak{S}_2$-symmetry, the case (iii) is excluded as well. It follows that $\deg M = \deg N = \deg P = \{0, 1\}$.

$\square$

7 Determination of $\mathcal{G}_{\mathbf{W}}(M \times N, P)$

Let M, N and $P \in \mathcal{S}$. In this section, we will compute the space $\mathcal{G}_{\mathbf{W}}(M \times N, P)$.

We will always assume that $\operatorname{Supp} P = \operatorname{Supp} M + \operatorname{Supp} N$, otherwise it is obvious that $\mathcal{G}_{\mathbf{W}}(M \times N, P) = 0$. In the previous section, it has been shown that $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) \leq 2$, and the case $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) = 2$ has been determined. So it remains to decide when $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ is zero or not.

The final result is very simple to state, because $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P)$ only depends on $\deg M, \deg N$ and $\deg P$.

7.1 Upper bound for $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P)$

Lemma 23. *Let M, N, P and $Q \in \mathcal{S}$ and let $\phi \in \mathcal{G}_{\mathfrak{sl}(2)}(P, Q)$. We have*

$$\phi_* \tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(M \times N, P) \subset \tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(M \times N, Q)$$

Proof. Step 1: Set $\Omega_1 = L_0^2 + L_0 - L_{-1}L_1$ and $\Omega_2 = L_0^2 + 2L_0 - L_{-2}L_2$. Indeed Ω_1 is the Casimir element of $U(\mathfrak{sl}(2))$ and it acts as some scalar $c(X)$ on any $\mathbf{W}$-module $X \in \mathcal{S}$. It turns out that Ω_2 acts on X as $4c(X)$.

Step 2: In order to prove the lemma, we can assume that $\phi \neq 0$. Therefore $c(P) = c(Q)$ and Ω_2 acts by the same scalar on P and on Q. Thus we get

$$\Omega_2 \circ \phi = \phi \circ \Omega_2.$$

Since ϕ commutes with L_0, we get

$$(L_{-2}L_2) \circ \phi = \phi \circ (L_{-2}L_2).$$

from which the lemma follows. $\square$

Lemma 24. *Let δ_1, δ_2 and $\gamma \in \mathbb{C}$ and let u and v be $\mathbb{Z}$-cosets. Assume that none of the following conditions is satisfied*

(i) $\gamma = 0, 1/2$ *or* 1,

(ii) $\delta_1 = -1/2$, $\delta_2 \in \{0, 1\}$ *and* $\gamma \in \{-1/2, 3/2\}$,

(iii) $\delta_1 \in \{0, 1\}$, $\delta_2 = -1/2$ *and* $\gamma \in \{-1/2, 3/2\}$.

Then we have:

$$\dim \mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma}) + \dim \mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{1-\gamma}) \leq 1.$$

Proof. By Lemma 4, there exists an isomorphism $\phi : \mathcal{G}_{\mathfrak{sl}(2)}(\Omega_{u+v}^{1-\gamma}) \to \mathcal{G}_{\mathfrak{sl}(2)}(\Omega_{u+v}^{\gamma})$. Obviously we have $\mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma}) \subset \tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma})$ and by the previous lemma we also have $\phi_* \mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{1-\gamma}) \subset \tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma})$.

By condition (i) and Lemma 6, the two subspaces $\mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma})$ and $\phi_* \mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{1-\gamma})$ intersects trivially, thus we have

$$\dim \mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma}) + \dim \mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{1-\gamma}) \leq \dim \tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma}).$$

By conditions (i), (ii) and (iii), Lemma 21, we have $\dim \tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma}) \leq 1$, which proves the lemma.

□

7.2 Necessary condition for $\tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma}) \neq 0$

Recall the notations of the previous section. For $k = 1$, 2, set

$$\begin{aligned}
a_k(x,y) &= (x + k\delta_1)(y - k\delta_2),\\
b_k(x,y) &= k^2(\delta_1 + \delta_2 - \gamma - \delta_1^2 - \delta_2^2 + \gamma^2) - 2xy,\\
c_k(x,y) &= (x - k\delta_1)(y + k\delta_2).
\end{aligned}$$

Given an auxiliary integer l, set $A_{il}(x,y) = a_i(x+l, y-l)$, $B_{il}(x,y) = b_i(x+l, y-l)$ and $C_{il}(x,y) = c_i(x+l, y-l)$ and set

$$\mathbf{M} = \begin{pmatrix}
A_{1,5}(x,y) & B_{1,5}(x,y) & C_{1,5}(x,y) & 0 & 0 & 0\\
0 & A_{1,4}(x,y) & B_{1,4}(x,y) & C_{1,4}(x,y) & 0 & 0\\
0 & 0 & A_{1,3}(x,y) & B_{1,3}(x,y) & C_{1,3}(x,y) & 0\\
0 & 0 & 0 & A_{1,2}(x,y) & B_{1,2}(x,y) & C_{1,2}(x,y)\\
A_{2,4}(x,y) & 0 & B_{2,4}(x,y) & 0 & C_{2,4}(x,y) & 0\\
0 & A_{2,3}(x,y) & 0 & B_{2,3}(x,y) & 0 & C_{2,3}(x,y)
\end{pmatrix},$$

Moreover set

$$\mathbf{D}_{\delta_1,\delta_2,\gamma}(x,y) = \det \mathbf{M}.$$

In what follows, we will consider $\mathbf{D}_{\delta_1,\delta_2,\gamma}(x,y)$ as a polynomial in the variables x and y, with parameters δ_1, δ_2 and γ.

Lemma 25. *If $\tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma}) \neq 0$ then $\mathbf{D}_{\delta_1,\delta_2,\gamma}(x,y) = 0$, for all $(x,y) \in \mathbb{C}^2$.*

Proof. Assume that $\mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma}) \neq 0$. Thus there exists a L_0-equivariant bilinear map $\pi : \Omega_u^{\delta_1} \times \Omega_v^{\delta_2} \to \Omega_{u+v}^{\gamma}$ whose germ is a non-zero element of $\tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma})$. For $(x,y) \in u \times v$, define the scalar $X(x,y)$ by the identity

$$\pi(e_x^{\delta_1}, e_y^{\delta_2}) = X(x,y) e_{x+y}^{\gamma}.$$

Set $\mathbf{X}(x,y) = (X(x+6, y-6), X(x+5, y-5), \ldots, X(x+1, y-1))$. Using Lemmas 18 and 20 there exists (x_0, y_0) such that

(i) $(X(x+2, y-2), X(x+1, y-1)) \neq 0$, and

(ii) $\mathbf{M}.{}^t\mathbf{X}(x,y) = 0$,

for all $(x,y) \in C(x_0, y_0)$. The first assertion implies that $\mathbf{X}(x,y) \neq 0$ for any $(x,y) \in C(x_0,y_0)$. Thus $\det \mathbf{M}$ vanishes on $C(x_0, y_0)$. Since $C(x_0, y_0)$ is Zariski dense, $\mathbf{D}_{\delta_1,\delta_2,\gamma}(x,y) = 0$, for all $(x,y) \in \mathbb{C}^2$.

□

7.3 Zeroes of the polynomials $p_{i,j}(\delta_1, \delta_2, \gamma)$

Define the polynomials $p_{i,j}(\delta_1, \delta_2, \gamma)$ by the identity

$$\mathbf{D}_{\delta_1,\delta_2,\gamma}(x,y) = \textstyle\sum_{i,j} p_{i,j}(\delta_1, \delta_2, \gamma) x^i y^j.$$

Since the entries of the matrix $\mathbf{M}$ are quadractic polynomials in x, y, δ_1, δ_2 and γ, $\mathbf{D}_{\delta_1,\delta_2,\gamma}(x,y)$ is a polynomial of degree ≤ 12. Set

$$C(\delta_1, \delta_2, \gamma) = (\delta_1 + \delta_2 + \gamma)(\delta_1 + \delta_2 - \gamma)(\delta_1 + \delta_2 + 1 - \gamma)(\delta_1 + \delta_2 - 1 + \gamma).$$

Lemma 26. *(i) We have $p_{i,j}(\delta_1, \delta_2, \gamma) = p_{i,j}(\delta_1, \delta_2, 1-\gamma)$,*

(ii) Each polynomial $p_{i,j}(\delta_1, \delta_2, \gamma)$ is divisible by $C(\delta_1, \delta_2, \gamma)$.

Proof. All entries of the matrix $\mathbf{M}$ are invariant under the involution $\gamma \mapsto 1-\gamma$, so we have

$$\mathbf{D}_{\delta_1,\delta_2,\gamma}(x,y) = \mathbf{D}_{\delta_1,\delta_2,1-\gamma}(x,y)$$

which implies the first assertion.

It follows from Lemma 11 that $\mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma}) \neq 0$ whenever $\gamma = \delta_1 + \delta_2$ or $\gamma = \delta_1 + \delta_2 + 1$.

Hence by Lemma 25, as a polynomial in $\delta_1, \delta_2, \gamma, x$ and y, $\mathbf{D}_{\delta_1,\delta_2,\gamma}(x,y)$ is divisible by $D(\delta_1, \delta_2, \gamma)$, where $D(\delta_1, \delta_2, \gamma) = (\delta_1 + \delta_2 - \gamma)(\delta_1 + \delta_2 + 1 - \gamma)$. By the first assertion, it is also divisible by $D(\delta_1, \delta_2, 1-\gamma)$. Since we have

$$C(\delta_1, \delta_2, \gamma) = D(\delta_1, \delta_2, \gamma) D(\delta_1, \delta_2, 1-\gamma)$$

each $p_{i,j}$ is divisible $C(\delta_1, \delta_2, \gamma)$. □

Denote by τ the involution $(\delta_1, \delta_2, \gamma) \mapsto (\delta_1, \delta_2, 1-\gamma)$. Let $\mathfrak{Z} \subset \mathbb{C}^3$ be the following set

$\mathfrak{Z} = (\bigcup_{0 \le i \le 1} H_i \cup H_i^\tau) \bigcup (\bigcup_{1 \le i \le 4} D_i \cup D_i^\tau) \bigcup (\bigcup_{1 \le i \le 2} \{P_i \cup P_i^\tau)\}$,

where the planes H_i, the lines D_i and the points P_i are defined in Section 4.5. For a polynomial f, denote by $Z(f)$ its zero set.

Lemma 27. *We have $Z(p_{1,3}) \cap Z(p_{3,1}) \cap Z(p_{2,2}) \subset \mathfrak{Z}$.*

The proof requires the explicit computation of $\mathbf{D}_{\delta_1,\delta_2,\gamma}(x, y) = \det \mathbf{M}$, and we have used MAPLE for this purpose. The next proof contains the explicit expressions of $p_{1,3}, p_{3,1}$ and $p_{2,2}$. The whole expression for $\det \mathbf{M}$ is given in Appendix A.

Proof. Step 1: By Lemma 26, there are polynomials q_{ij} such that

$$p_{ij}(\delta_1, \delta_2, \gamma) = C(\delta_1, \delta_2, \gamma) q_{ij}(\delta_1, \delta_2, \gamma).$$

Since $Z(C)$ is the union of the four planes $H_0, H_0^\tau, H_1, H_1^\tau$, it remains to prove that $Z(q_{1,3}) \cap Z(q_{3,1}) \cap Z(q_{2,2}) \subset \mathfrak{Z}$.

Using MAPLE, it turns out that

$$\begin{aligned} q_{2,2} =& \gamma^2(1-\gamma)^2 + 2\gamma(1-\gamma)(\delta_1^2 + \delta_2^2 - 2\delta_1\delta_2 - 2(\delta_1 + \delta_2) + 4) \\ &+ (\delta_1^4 + \delta_2^4 - 4\delta_1\delta_2(\delta_1^2 + \delta_2^2) + 38\delta_1^2\delta_2^2) - 4(\delta_1 + \delta_2)^3 \\ &- (13(\delta_1^2 + \delta_2^2) - 6\delta_1\delta_2) + 4(\delta_1 + \delta_2) + 12, \end{aligned}$$

$$-\frac{1}{8} q_{1,3} = \delta_1(\delta_1 - 1)[\gamma(1-\gamma) + \delta_1^1 + \delta_2^2 - 4\delta_1\delta_2 + 3(\delta_1 - \delta_2) + 2].$$

$$-\frac{1}{8} q_{3,1} = \delta_2(\delta_2 - 1)[\gamma(1-\gamma) + \delta_1^1 + \delta_2^2 - 4\delta_1\delta_2 - 3(\delta_1 - \delta_2) + 2],$$

Step 2: The previous expressions provide (miraculous) factorizations

$$-\tfrac{1}{8} q_{1,3} = L_1 L_2 Q \text{ and } -\tfrac{1}{8} q_{3,1} = L_1' L_2' Q'$$

where L_1, L_2, L_1', and L_2' are degree one polynomials and Q and Q' are quadratic polynomials. We have to prove that $Z(P) \cap Z(P') \cap Z(q_{2,2}) \subset \mathfrak{Z}$ for any factor P of $q_{1,3}$ and any factor P' of $q_{3,1}$. This amonts to 9 cases, which will be treated seperately.

Step 3: proof that $Z(L_i) \cap Z(L_j') \cap Z(q_{2,2}) \subset \mathfrak{Z}$, $\forall i, j \in \{1, 2\}$.

We claim that, in each case, the intersection consists of 4 points lying in $\mathfrak{Z}$. Since the four cases are similar, we will only consider the case where the first factor is δ_1 and the second one is δ_2.

For a point $(0,0,\gamma) \in Z(\delta_1) \cap Z(\delta_2) \cap Z(q_{2,2})$, we have

$$q_{2,2}(0,0,\gamma) = \gamma^2(1-\gamma)^2 + 8\gamma(1-\gamma) + 12{=}0.$$

Thus we have $\gamma(1-\gamma) = -2$ or -6. It follows that $Z(\delta_1) \cap Z(\delta_2) \cap Z(p_{2,2})$ consists of the four points $(0,0,-2), (0,0,-1), (0,0,2), (0,0,3)$ which are all in $\mathfrak{Z}$.

Step 4: proof that $Z(L_i) \cap Z(Q') \cap Z(q_{2,2}) \subset \mathfrak{Z}$, $\forall i \in \{1,2\}$.

More precisely, we claim that the planar quadric $Z(L_i) \cap Z(Q')$ consists of two lines which are both in $\mathfrak{Z}$. Since the two cases are similar, we will just treat the case where the factor L_i is δ_1. We have

$$\begin{aligned} Q'(0,\delta_2,\gamma) &= \gamma(1-\gamma) + \delta_2^2 + 3\delta_2 + 2 \\ &= -(\gamma + \delta_2 + 1)(\gamma - \delta_2 - 2), \end{aligned}$$

which proves the claim. It follows that $Z(L_i) \cap Z(Q') \cap Z(q_{2,2}) \subset \mathfrak{Z}$.

Step 5: Proof that $Z(Q) \cap Z(L'_j) \cap Z(q_{2,2}) \subset \mathfrak{Z}$, $\forall j \in \{1,2\}$.

This case is identical to the previous one.

Step 6: Proof that $Z(Q) \cap Z(Q') \cap Z(q_{2,2}) \subset \mathfrak{Z}$.

Indeed $Q' - Q$ is a scalar multiple of $\delta_1 - \delta_2$ and therefore $Z(Q) \cap Z(Q')$ is (again miraculously) a planar quadric.

We have $Q(\delta,\delta,\gamma) = \gamma(1-\gamma) - 2\delta^2 + 2$, so $Z(Q) \cap Z(Q')$ is the sets of all $(\delta,\delta,\gamma) \in \mathbb{C}^3$ such that $\gamma(1-\gamma) = 2\delta^2 - 2$. Since $q_{2,2}(\delta,\delta,\gamma)$ is a polynomial in δ and $\gamma(1-\gamma)$ we can eliminate $\gamma(1-\gamma)$. We have

$$q_{2,2}(\delta,\delta,\gamma) = 12\delta(3\delta+2)(\delta-1)^2$$

for any $(\delta,\delta,\gamma) \in Z(Q) \cap Z(Q')$. It follows that $Z(Q) \cap Z(Q') \cap Z(q_{2,2})$ consists of the 6 points $(1,1,0), (1,1,1), (0,0,-1), (0,0,2), (-2/3,-2/3,-2/3)$ and $(-2/3,-2/3,5/3)$. Since there are all in $\mathfrak{Z}$, the proof is complete.

$\square$

With more care, it is easy to prove that $\bigcap Z(p_{i,j})$ is precisely $\mathfrak{Z}$ but this is not necessary for what follows.

7.4 Determination of $\mathcal{G}_{\mathbf{W}}(M \times N, P)$

Recall that $\mathfrak{z}^*$ the set of all $(\delta_1,\delta_2,\gamma) \in \mathfrak{z}$ such that $\{\delta_1,\delta_2,\gamma\} \not\subset \{0,1\}$. Let M, N and P be in $\mathcal{S}$. In order to determine $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ we will always assume that

$$\operatorname{Supp} P = \operatorname{Supp} M + \operatorname{Supp} N.$$

Otherwise $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ would be obviously zero. Next let $\delta_1 \in \deg M$, $\delta_2 \in \deg N$ and $\gamma \in \deg P$.

Theorem 2. *We have*

(i) $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) = 2$ *if* $\{\delta_1, \delta_2, \gamma\} \subset \{0, 1\}$*, and the maps* π_1*,* π_2 *of Lemma 11 form a basis of this space,*

(ii) $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) = 1$ *if* $(\delta_1, \delta_2, \gamma) \in \mathfrak{z}^*$ *and the map* π *of Table 1 provides a generator of this space,*

(iii) $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) = 0$ *otherwise.*

Proof. Set $u = \operatorname{Supp} M$ and $v = \operatorname{Supp} N$. By Lemmas 3 and 5, we can assume that $M = \Omega_u^{\delta_1}$, $N = \Omega_v^{\delta_2}$ and $P = \Omega_{u+v}^{\gamma}$.

Step 1: We claim that $(\delta_1, \delta_2, \gamma)$ belongs to $\mathfrak{z}$ if $\mathcal{G}_{\mathbf{W}}(M \times N, P) \neq 0$.

Assume that $\mathcal{G}_{\mathbf{W}}(M \times N, P) \neq 0$. Since $\tilde{\mathcal{G}}_{\mathfrak{sl}(2)}(M \times N, P) \neq 0$ it follows from Lemmas 25 and 27 that $(\delta_1, \delta_2, \gamma)$ belongs to $\mathfrak{Z}$. It is clear from its definition that $\mathfrak{Z} \subset \mathfrak{z} \cup \mathfrak{z}^{\tau}$. Hence $(\delta_1, \delta_2, \gamma)$ or $(\delta_1, \delta_2, 1-\gamma)$ belongs to $\mathfrak{z}$.

When $(\delta_1, \delta_2, 1-\gamma) \notin \mathfrak{z}$ the claim is proved. Moreover if $\gamma = 0, 1/2$ or 1, we have $\mathcal{G}_{\mathbf{W}}(\Omega_{u+v}^{\gamma}) = \mathcal{G}_{\mathbf{W}}(\Omega_{u+v}^{1-\gamma})$ and thus $\mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{1-\gamma}) = \mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma})$, which proves the claim in this case. Therefore, we can assume that $(\delta_1, \delta_2, 1-\gamma) \in \mathfrak{z}$ and that $\gamma \notin \{0, 1/2, 1\}$.

By Lemma 11, we have $\mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{1-\gamma}) \neq 0$. Therefore it follows that

$$\dim \mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma}) + \dim \mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{1-\gamma}) \geq 2.$$

By Lemma 24 we have

(i) $\delta_1 = -1/2$, $\delta_2 \in \{0, 1\}$ and $\gamma \in \{-1/2, 3/2\}$, or

(ii) $\delta_1 \in \{0, 1\}$, $\delta_2 = -1/2$, and $\gamma \in \{-1/2, 3/2\}$.

These 8 possible triples for $(\delta_1, \delta_2, \gamma)$ belong to $\mathfrak{z}$ and therefore the claim is proved.

Step 2: Assertion (i) follows from Lemma 22. From now on, we can assume that $\{\delta_1, \delta_2, \gamma\} \not\subset \{0, 1\}$. It follows that $\dim \mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma}) = 0$ or 1. In particular Assertion (ii) and (iii) are equivalent and it is enough to prove the first one.

If $(\delta_1, \delta_2, \gamma) \in \mathfrak{z}^*$ we have $\mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma}) \neq 0$ by Lemma 11 and therefore $\dim \mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma}) = 1$. Conversely if $\dim \mathcal{G}_{\mathbf{W}}(\Omega_u^{\delta_1} \times \Omega_v^{\delta_2}, \Omega_{u+v}^{\gamma}) = 1$ it follows from the previous step that $(\delta_1, \delta_2, \gamma)$ belongs to $\mathfrak{z}^*$. Thus assertion (ii) is proved. $\square$

8 On the map $\mathbf{B_W}(M \times N, P) \to \mathcal{G}_{\mathbf{W}}(M \times N, P)$

Let M, N and P be in $\mathcal{S}$. The space $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ has been determined by Theorem 2. In particular $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ has always dimension 0, 1 or 2. In the Sections 8-10, we determine which germs $\mu \in \mathcal{G}_{\mathbf{W}}(M \times N, P)$ can be lifted to a $\mathbf{W}$-equivariant bilinear map $\pi : M \times N \to P$. Since the final result contains many particular case, it has been split into two parts. Indeed Theorem 3.1 (in Section 9) involves the case where $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ has dimension one, and and Theorem 3.2 (in Section 10) involves the case where $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ has dimension two. In this section, we recall general facts and conventions used in Sections 9 and 10.

8.1 Germs and $\mathfrak{S}_3$-symmetry

Let $M, N, P \in \mathcal{S}$. Recall the exact sequence:

$$0 \to \mathbf{B}^0_{\mathbf{W}}(M \times N, P) \to \mathbf{B_W}(M \times N, P) \to \mathcal{G}_{\mathbf{W}}(M \times N, P).$$

Determining the image of the map $\mathbf{B_W}(M \times N, P) \to \mathcal{G}_{\mathbf{W}}(M \times N, P)$ is easy, but it requires a very long case-by-case analysis. It would be pleasant to use the $\mathfrak{S}_3$-symmetry to reduce the number of cases. Unfortunately the definition of $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ is not $\mathfrak{S}_3$-symmetric. However, set

$$\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P) = \mathbf{B_W}(M \times N, P)/\mathbf{B}^0_{\mathbf{W}}(M \times N, P).$$

Lemma 28. *For any M, N and $P \in \mathcal{S}$, we have*

$$\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P^*) = \overline{\mathbf{B}}_{\mathbf{W}}(M \times P, N^*)$$

Proof. By Lemma 9, the space $\mathbf{B}^0_{\mathbf{W}}(M \times N, P^*)$ is exactly the space of degenerate $\mathbf{W}$-equivariant maps from $M \times N$ to P^*. Hence $\mathbf{B}^0_{\mathbf{W}}(M \times N, P^*)$ and $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P^*)$ are fully symmetric in M, N and P. $\square$

8.2 List of cases for the proof of Theorem 3

Start with a general result:

Lemma 29. *Let M, N and $P \in \mathcal{S}$ be irreducible $\mathbf{W}$-modules. We have*

$$\mathbf{B_W}(M \times N, P) \simeq \mathcal{G}_{\mathbf{W}}(M \times N, P).$$

Proof. Looking at Table 2, it is clear that $\mathbf{B}^0_{\mathbf{W}}(M \times N, P) = 0$ whenever M, N and P are irreducible. Moreover, it is clear from Table 1 that any germ can be lifted. $\square$

Let M, N and $P \in \mathcal{S}$. In order to determine $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$, we will always tacitely assume that $\mathcal{G}_{\mathbf{W}}(M \times N, P) \neq 0$. By the previous lemma, we can assume that at least one module is reducible, i.e. in the AB-family. As usual, we will assume that all modules are indecomposable. Using the $\mathfrak{S}_3$-symmetry, we can reduce to the following 6 cases:

1. $\deg M = \deg N = \{0, 1\}$ and $\deg P = 2$,
2. $\deg M = \deg N = \{0, 1\}$ and $\deg P = 3$,
3. $\deg M = \{0, 1\}$, $\deg N = \delta$ and $\deg P = \delta$ with $\delta \in \mathbb{C} \setminus \{0, 1\}$.
4. $\deg M = \{0, 1\}$, $\deg N = \delta$ and $\deg P = \delta + 1$ with $\delta \in \mathbb{C} \setminus \{0, 1\}$.
5. $\deg M = \{0, 1\}$, $\deg N = \delta$ and $\deg P = \delta + 2$ with $\delta \in \mathbb{C} \setminus \{0, 1\}$
6. $\deg M = \deg N = \deg P = \{0, 1\}$.

The cases case 1-5 are treated in Section 9. In this case, we have $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) = 1$, so it is enough to decide if $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$ is zero or not. The case 6 is treated in section 10. In this case, $\mathcal{G}_{\mathbf{W}}(M \times N, P)$ is two dimensional and the analysis is more involved.

8.3 Typical argument for the proof of Theorem 3

Let $M \in \mathcal{S}$ and let u be its support. In Sections 9 and 10, we will denote by $(e_x^M)_{x \in u}$ a basis of M as in Section 1.

The proofs of Theorems 3.1 and 3.2 are given by several lemmas and a repeated procedure, that we call an *argument by restriction*, which is described as follows.

For an integer $d \in \mathbb{Z}_{>1}$, the subalgebra $\mathbf{W}^{(d)} := \bigoplus_n \mathbb{C} L_{dn}$ is isomorphic to $\mathbf{W}$. Let M be a $\mathbf{W}$-module in the class $\mathcal{S}$ and let $x \in \mathrm{Supp}\, M$. The subspace

$$M_d(x) := \bigoplus_{\substack{y \in u \\ x - y \in d\mathbb{Z}}} M_y$$

of M is a $\mathbf{W}^{(d)}$-module. Moreover, when $x \notin d\mathbb{Z}$, $M_d(x)$ is irreducible.

Now, let M, N and P be $\mathbf{W}$-modules in the class $\mathcal{S}$, let $x \in \mathrm{Supp}\, M$, $y \in \mathrm{Supp}\, N$ and let $\overline{\pi} \in \mathcal{G}_{\mathbf{W}}(M \times N, P)$. Since $\mathcal{G}_{\mathbf{W}^{(d)}}(X \times Y, Z) \simeq \mathbf{B}_{\mathbf{W}^{(d)}}(X \times Y, Z)$ whenever X, Y and Z are irreducible $\mathbf{W}^{(d)}$-modules of the class $\mathcal{S}$, $\overline{\pi}$ has unique lifting π to $M_d(x) \times N_d(y)$ whenever $x, y, x+y \notin d\mathbb{Z}$. Hence, varying d and x, y, we see that $\pi(e_x^M, e_y^N)$ is uniquely determined by $\overline{\pi}$ whenever $x, y, x+y \neq 0$.

9 Computation of $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$ when $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) = 1$

9.1 The Theorem 3.1

The following table provides a list of triples (M, N, P) of $\mathbf{W}$-modules of the class $\mathcal{S}$ and one non-zero element $\pi \in \overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$. Since for each entry (M, N, P) of Table 3 we have $\deg M$ or $\deg N$ or $\deg P \neq \{0, 1\}$, the corresponding π is a basis of $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$.

In what follows, we denote by d_ξ and d^ξ the natural maps:

$$d^\xi : \Omega_0^0 \to A_\xi \text{ and } d_\xi : B_\xi \to \Omega_0^1.$$

Conversely, we have

Theorem 3.1 *Let M, N and P be $\mathbf{W}$-modules of the class $\mathcal{S}$ with $\deg M$ or $\deg N$ or $\deg P \neq \{0, 1\}$. Then $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$ has dimension one if the triple (M, N, P) appears in Table 3. Otherwise, we have $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P) = 0$.*

9.2 The case $\deg M = \deg N = \{0, 1\}$ and $\deg P = 2$

In this case, there can be five subcases as follows:

1. $(M, N, P) = (A_\xi, \Omega_u^0, \Omega_u^2)$ with $u \not\equiv 0 \bmod \mathbb{Z}$,
2. $(M, N, P) = (B_\xi, \Omega_u^0, \Omega_u^2)$ with $u \not\equiv 0 \bmod \mathbb{Z}$,
3. $(M, N, P) = (A_\eta, A_\xi, \Omega_0^2)$,
4. $(M, N, P) = (A_\eta, B_\xi, \Omega_0^2)$,
5. $(M, N, P) = (B_\eta, B_\xi, \Omega_0^2)$.

Lemma 30. *Let M, N, P be $\mathbf{W}$-modules of the class $\mathcal{S}$ as above. Then $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$ is trivial iff $(M, N, P) = (A_\eta, A_\xi, \Omega_0^2)$ with $\xi \neq \infty$ and $\eta \neq \infty$*

Table 3: **Non-zero** $\overline{\mathbf{B}}_{\mathbf{W}}(M\times N,P)$**, when** $\deg M$, $\deg N$ **or** $\deg P\neq\{0,1\}$

	$M\times N$ or $N\times M$	P	π
1.	$\Omega_u^{\delta_1}\times\Omega_v^{\delta_2}$	$\Omega_{u+v}^{\delta_1+\delta_2}$	$P_{u,v}^{\delta_1,\delta_2}$
2.	$\Omega_u^{\delta_1}\times\Omega_v^{\delta_2}$	$\Omega_{u+v}^{\delta_1+\delta_2+1}$	$B_{u,v}^{\delta_1,\delta_2}$
3.	$A_\xi\times\Omega_u^{\delta}$	$\Omega_u^{\delta+1}$	$B_{0,u}^{0,\delta}(\xi)$
4.	$\Omega_u^{\delta}\times\Omega_{-u}^{-\delta}$	B_ξ	$B_{u,-u}^{\delta,-\delta}(\xi)$
5.	$\Omega_u^{-2/3}\times\Omega_v^{-2/3}$	$\Omega_{u+v}^{5/3}$	$G_{u,v}$
6.	$B_\xi\times\Omega_u^{\delta}$	$\Omega_u^{\delta+1}$	$P_{0,u}^{1,\delta}\circ(d_\xi\times id)$
7.	$\Omega_u^{\delta}\times\Omega_{-u}^{-\delta}$	A_ξ	$d^\xi\circ P_{u,-u}^{\delta,-\delta}$
8.	$B_\xi\times\Omega_u^{\delta}$	$\Omega_u^{\delta+2}$	$B_{0,u}^{1,\delta}\circ(d_\xi\times id)$
9.	$\Omega_u^{\delta}\times\Omega_{-u}^{-\delta-1}$	A_ξ	$d^\xi\circ B_{u,-u}^{\delta,-\delta-1}$
10.	$A_\eta\times B_\xi$	Ω_0^2	$B_{0,0}^{0,1}(\eta)\circ(id\times d_\xi)$
11.	$A_\eta\times\Omega_0^{-1}$	A_ξ	$d^\xi\circ B_{0,0}^{0,-1}(\eta)$
12.	$B_\xi\times\Omega_0^{-1}$	B_η	$B_{0,0}^{1,-1}(\eta)(d_\xi\times id)$
13.	$B_\eta\times B_\xi$	Ω_0^2	$P_{0,0}^{1,1}\circ(d_\eta\times d_\xi)$
14.	$B_\eta\times\Omega_0^{-1}$	A_ξ	$d^\xi\circ P_{0,0}^{1,-1}\circ(d_\eta\times id)$
15.	$B_\eta\times B_\xi$	Ω_0^3	$B_{0,0}^{1,1}\circ(d_\eta\times d_\xi)$
16.	$B_\eta\times\Omega_0^{-2}$	A_ξ	$d^\xi\circ B_{0,0}^{1,-2}\circ(d_\eta\times id)$

The degree condition implies the following restrictions: $\{\delta_1,\delta_2,\delta_1+\delta_2\}\not\subset\{0,1\}$ in line 1, $(\delta_1,\delta_2)\neq(0,0)$ in line 2, $\delta\neq 0$ in lines 3 and 4, and $\delta\neq 0,\ 1$ in lines 6 and 7.

Proof. By Table 3, except for the case 3 with $\eta=\infty$ or $\xi=\infty$, we have $\dim\overline{\mathbf{B}}_{\mathbf{W}}(M\times N,P)=1$. Hence, we show the proposition in the case $(M,N,P)=(A_\eta,A_\xi,\Omega_0^2)$.

Let (a,b) and (c,d) be projective coordinates of $\eta,\xi\in\mathbb{P}^1$, respectively, and let $\{e_m^M\}$, $\{e_m^N\}$ and $\{e_m^P\}$ be basis of $A_{a,b}$, $A_{c,d}$ and Ω_0^2, respectively, as in Section 1.1. Assume that $\overline{\mathbf{B}}_{\mathbf{W}}(M\times N,P)\neq 0$. By an argument by restriction, one sees that there exists $\pi\in\mathbf{B}_{\mathbf{W}}(M\times N,P)$ such that $\pi(e_m^M,e_n^N)=e_{m+n}^P$ whenever $m,n,m+n\neq 0$. It can be checked that this formula extends to the case $m,n\neq 0$. Set $\pi(e_m^M,e_0^N)=X_1(m)e_m^P$ and $\pi(e_0^M,e_m^N)=X_2(m)e_m^P$ for $m\neq 0$. Calculating $L_n.\pi(e_m^M,e_0^N)$ for $n=1,2$, one obtains

$$(m+2n)X_1(m)=(m+n)X_1(m+n)+cn^2+dn$$

from which we have $X_1(m)=-cm+d$. Similarly, one also obtains $X_2(m)=$

$-am + b$. By calculating $L_m.\pi(e_0^M, e_0^N)$, we obtain $ac = 0$. $\square$

9.3 The case $\deg M = \deg N = \{0, 1\}$ and $\deg P = 3$

In this case, there can be five subcases as follows:

1. $(M, N, P) = (A_\xi, \Omega_u^0, \Omega_u^3)$ with $u \not\equiv 0 \bmod \mathbb{Z}$,
2. $(M, N, P) = (B_\xi, \Omega_u^0, \Omega_u^3)$ with $u \not\equiv 0 \bmod \mathbb{Z}$,
3. $(M, N, P) = (A_\xi, A_\eta, \Omega_0^3)$,
4. $(M, N, P) = (A_\xi, B_\eta, \Omega_0^3)$,
5. $(M, N, P) = (B_\xi, B_\eta, \Omega_0^3)$.

Lemma 31. *Let M, N, P be $\mathbf{W}$-modules as above. Then $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$ is trivial iff (M, N, P) is one of the following types:*

1. $(M, N, P) = (A_\xi, \Omega_u^1, \Omega_u^3)$ *with* $\xi \neq \infty$ *and* $u \not\equiv 0 \bmod \mathbb{Z}$,
2. $(M, N, P) = (A_\xi, A_\eta, \Omega_0^3)$ *with* $(\xi, \eta) \neq (\infty, \infty)$,
3. $(M, N, P) = (A_\xi, B_\eta, \Omega_0^3)$ *with* $\xi \neq \infty$.

Proof. By Table 3, it is clear that $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$ is trivial only if (M, N, P) is one of the three cases in Lemma 31. Hence, it is sufficient to show that, for these three cases, $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$ is trivial.

The proof for the first and third cases are similar to that of Lemma 30 and is left to the reader. The second case can be proved as follows.

Choose any $\tau \in \mathbb{P}^1$. Any non-degenerate $\pi \in \mathbf{B}_{\mathbf{W}}(A_\eta \times A_\xi, \Omega_0^3)$ induces a non-degenerate bilinear map $\pi' \in \mathbf{B}_{\mathbf{W}}(A_\eta \times B_\tau, \Omega_0^3)$, by composing with the map $B_\tau \to \overline{A} \hookrightarrow A_\xi$. It follows from the first case that $\eta = \infty$. Similarly, we have $\xi = \infty$. $\square$

9.4 The case $\deg M = \{0,1\}$, $\deg N = \delta$ and $\deg P = \delta + 2$ with $\delta \in \mathbb{C} \setminus \{0,1\}$

In this case, there can be two subcases as follows:

1. $(M, N, P) = (A_\xi, \Omega_u^\delta, \Omega_u^{\delta+2})$,
2. $(M, N, P) = (B_\xi, \Omega_u^\delta, \Omega_u^{\delta+2})$.

Lemma 32. *Let M, N, P be $\mathbf{W}$-modules as above. Then, $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$ is trivial iff $M = A_\xi$ with $\xi \neq \infty$ and $\delta \neq -1$.*

The proof is similar to that of Lemma 30.

9.5 The case $\deg M = \{0,1\}$, $\deg N = \delta$ and $\deg P = \delta + 1$ with $\delta \in \mathbb{C} \setminus \{0,1\}$

In this case, there can be two subcases as follows:

1. $(M, N, P) = (A_\xi, \Omega_u^\delta, \Omega_u^{\delta+1})$,
2. $(M, N, P) = (B_\xi, \Omega_u^\delta, \Omega_u^{\delta+1})$.

Looking at the lines 3 and 6 in Table 3, we obtain

Lemma 33. *Let M, N, P be $\mathbf{W}$-modules of the class $\mathcal{S}$ as above. Then $\dim \overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P) = 1$.*

9.6 The case $\deg M = \{0,1\}$, $\deg N = \delta$ and $\deg P = \delta$ with $\delta \in \mathbb{C} \setminus \{0,1\}$

In this case, there can be two subcases as follows:

1. $(M, N, P) = (A_\xi, \Omega_u^\delta, \Omega_u^\delta)$,
2. $(M, N, P) = (B_\xi, \Omega_u^\delta, \Omega_u^\delta)$.

Lemma 34. *Let M, N, P be $\mathbf{W}$-modules as above. Then, $\dim \overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P) = 1$ iff $M = B_\infty$.*

The proof is similar to that of Lemma 30.

10 Computation of $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$ when $\dim \mathcal{G}_{\mathbf{W}}(M \times N, P) = 2$

10.1 The Theorem 3.2

The following table provides a list of triples $(M,\ N,\ P)$ of **W**-modules of the class $\mathcal{S}$ and some elements $\pi \in \overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$. For each entry $(M,\ N,\ P)$ we have $\deg M = \deg N = \deg P = \{0,1\}$

Table 4: **The non-zero** $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$**, with** $\deg M = \deg N = \deg P = \{0,1\}$

	$M \times N$or $N \times M$	P	Elements of $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$
1.	$\Omega^0_u \times \Omega^0_v$	Ω^1_{u+v}	$P^{0,0}_{u,v} \circ (d \times id)$ and $P^{0,0}_{u,v} \circ (id \times d)$
2.	$\Omega^0_u \times \Omega^0_{-u}$	Ω^0_0	$P^{0,0}_{u,-u}$
3.	$\Omega^0_u \times \Omega^1_0$	Ω^1_u	$P^{0,1}_{u,0}$
4.	$\Omega^0_u \times \Omega^0_{-u}$	A_ξ	$d^\xi \circ P^{0,0}_{u,-u}$
5.	$B_\xi \times \Omega^0_u$	Ω^1_u	$P^{1,0}_{0,u} \circ (d_\xi \times id)$

To avoid repetitions, one can assume that $\xi \neq \infty$ in lines 4 or 5

From the table, it is clear that

(i) If $(M,\ N,\ P)$ appears in line 1 of the Table 4, the two listed elements of $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$ are linearly independent and therefore $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$ has dimension 2,

(ii) if $(M,\ N,\ P)$ appears in lines 2-5 of the Table 4, the listed element of $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$ is not zero and therefore $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P)$ has dimension ≥ 1.

Indeed, we have

Theorem 3.2 *Let M, N and P be* **W***-modules of the class $\mathcal{S}$ with* $\deg M = \deg N = \deg P = \{0,1\}$. *Then*

(i) if $(M,\ N,\ P)$ appears in line 1 of the Table 4, we have $\dim \overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P) = 2$,

(ii) if $(M,\ N,\ P)$ appears in lines 2-5 of the Table 4, we have $\dim \overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P) = 1$,

(iii) otherwise, we have $\overline{\mathbf{B}}_{\mathbf{W}}(M \times N, P) = 0$.

In order to prove Theorem 3.2, note that the case where M, N and P are irreducible is already treated in Lemma 29. Thus, up to the $\mathfrak{S}_3$-symmetry, there are only three cases to consider:

1. $(M, N) = (A_{\xi_1}, A_{\xi_2})$ and $P = A_\xi$ or $P = B_\xi$,
2. $(M, N) = (B_{\xi_1}, B_{\xi_2})$ and $P = A_{\xi_3}$ or $P = B_{\xi_3}$,
3. $(M, N) = (\Omega^0_u, \Omega^0_{-u})$ with $u \not\equiv 0 \bmod \mathbb{Z}$ and $P = A_\xi$ or $P = B_\xi$.

These cases will be treated in the next three subsections.

10.2 The case $(M, N) = (A_{\xi_1}, A_{\xi_2})$

Lemma 35. *Any bilinear map in $\mathbf{B_W}(A_{\xi_1} \times A_{\xi_2}, P)$, where P is in the AB-family, is degenerate.*

Proof. Any module P in the AB-family admits an almost-isomorphism to a module of the A-family. So we can assume $P \simeq A_{\xi_3}$ for some ξ_3 in $\mathbb{P}^1$.

Fix a basis $\{e^M_m\}$, $\{e^N_m\}$ and $\{e^P_m\}$ of $M = A_{\xi_1}$, $N = A_{\xi_2}$ and $P = A_{\xi_3}$, respectively, as in Section 1.1. We have $L_{-1}.e^M_0 \neq 0$ or $L_1.e^M_0 \neq 0$. Since both cases are similar, we case assume that $L_{-1}eu^M_0 \neq 0$. Using that $L_{-1}.e^N_1 = L_{-1}.e^P_1 = 0$, we obtain that $\pi(L_{-1}.e^M_0, e^N_1) = L_{-1}.\pi(e^M_0, e^N_1) - \pi(e^M_0, L_{-1}.e^N_1) = 0$, hence we have

$$\pi(e^M_{-1}, e^N_1) = 0.$$

Since $M_{\leq -1}$ (respectively $N_{\geq 1}$) is an irreducible Verma $\mathfrak{sl}(2)$-module (respectively the restricted dual of an irreducible Verma $\mathfrak{sl}(2)$-module), it follows that $M_{\leq -1} \otimes N_{\geq 1}$ is generated by $e^M_{-1} \otimes e^N_1$. Hence we get $\pi(M_{\leq -1} \times N_{\geq 1}) = 0$. Since $N_{\geq 1}$ is a $\mathbf{W}_{\geq 0}$-submodule and since $\overline{A}$ is the $\mathbf{W}_{\geq 0}$-submodule generated by $M_{\leq -1}$, we have

$$\pi(\overline{A} \times N_{\geq 1}) = 0,$$

from which it follows that π is degenerate. $\square$

10.3 The case $(M, N) = (B_{\xi_1}, B_{\xi_2})$

Recall that $d_\xi \circ P^{0,0}_{0,0}$ is a non-degenerate bilinear map from $B_\infty \times B_\infty \to A_\xi$, for any $\xi \in \mathbb{P}^1$. Let ξ_1, ξ_2 and $\xi_3 \in \mathbb{P}^1$, and let s be the number of indices i such that $\xi_i = \infty$.

By $\mathfrak{S}_3$ symmetry, $\overline{\mathbf{B}}_{\mathbf{W}}(B_{\xi_1} \times B_{\xi_2}, A_{\xi_3})$ is not zero $s \geq 2$. More precisely, we have:

Lemma 36. *We have*

(i) $\dim \overline{\mathbf{B}}_{\mathbf{W}}(B_{\xi_1} \times B_{\xi_2}, A_{\xi_3}) = s - 1$ *if* $s \geq 2$

(ii) $\dim \overline{\mathbf{B}}_{\mathbf{W}}(B_{\xi_1} \times B_{\xi_2}, B_{\xi_3}) = 1$ *if* $s = 3$

(iii) $\dim \overline{\mathbf{B}}_{\mathbf{W}}(B_{\xi_1} \times B_{\xi_2}, A_{\xi_3}) = \dim \overline{\mathbf{B}}_{\mathbf{W}}(B_{\xi_1} \times B_{\xi_2}, B_{\xi_3}) = 0$ *otherwise.*

Proof. Let $(a, b), (c, d)$ and (e, f) be projective coordinates of ξ_1, ξ_2 and $\xi_3 \in \mathbb{P}^1$, respectively, and fix a basis $\{e^M_m\}$, $\{e^N_m\}$ of $M = B_{a,b}$ and $N = B_{c,d}$, respectively, as in Section 1.1.

First, we consider $\pi \in \mathbf{B}_{\mathbf{W}}(B_{a,b} \times B_{c,d}, P)$ with $P = B_{e,f}$. Let $\{e^P_m\}$ be a basis of $B_{e,f}$ as in § 1.1. By calculating $L_{-n}.\pi(e^M_n, e^N_0)$ and $L_{-n}.\pi(e^M_0, e^N_n)$, we see that $\xi_1 = \xi_2 = \xi_3 \in \mathbb{P}^1$. Hence, we may assume that $(a, b) = (c, d) = (e, f)$ without loss of generality. Similarly, by calculating $L_m.\pi(e^M_n, e^N_0)$ and $L_m.\pi(e^M_0, e^N_n)$, we see that there exists a constant $C \in \mathbb{C}$ satisfying $\pi(e^M_m, e^N_0) = Ce^P_m = \pi(e^M_0, e^N_m)$ for any m. It can be shown that such a $\mathbf{W}$-equivariant map exists only if $a = 0$ or $C = 0$. In the former case, π is a scalar multiple of $P^{0,0}_{0,0}$. In the latter case, π factors through $\overline{A} \times \overline{A}$ and one can apply a similar argument to the latter half of the proof of Lemma 35 to see that π is degenerate.

Second, we consider $\pi \in B_{\mathbf{W}}(B_{a,b} \times B_{c,d}, P)$ with $P = A_{e,f}$. Let $\{e^P_m\}$ be a basis of $A_{e,f}$ as in § 1.1. By an argument by restriction, one sees that there exists constants $C_1, C_2 \in \mathbb{C}$ such that $\pi(e^M_m, e^N_n) = (C_1 m + C_2 n)e^P_{m+n}$ for $m, n, m + n \neq 0$. Set $\pi(e^M_m, e^N_0) = a(m)e^P_m, \pi(e^M_0, e^N_m) = b(m)e^P_m$ and $\pi(e^M_m, e^N_{-m}) = c(m)e^P_0$. It is clear that $a(0) = b(0) = c(0) = 0$. By calculating $L_{-n}.\pi(e^M_m, e^N_n)$ with $m, n, m + n \neq 0$, one obtains that $a(m) = -d^{-1}C_1 m$ if $c = 0$ and that $a(m) = C_1 = 0$ otherwise. Similarly, on obtains that $b(m) = -b^{-1}C_2 m$ if $a = 0$ and that $b(m) = C_2 = 0$ otherwise. Finally, by calculating $L_n.\pi(e^M_m, e^N_{-m})$ with $n \neq \pm m$, on obtains that $c(m) = -(C_1 - C_2)f^{-1}m$ if $e = 0$ and that $c(m) = 0$ and $C_1 = C_2$ otherwise. Hence, it follows that $\dim \overline{\mathbf{B}}_{\mathbf{W}}(B_{a,b} \times B_{c,d}, A_{e,f})$ is equal to 0 if $s \leq 1$ and is less than $s - 1$ if $s \geq 2$. Now, for $s \geq 2$, the result follows from Table 4. □

10.4 The case $(M, N) = (\Omega^0_u, \Omega^0_{-u})$ with $u \neq 0 \in \mathbb{C}/\mathbb{Z}$

The next lemma can be proved in a way similar to the proof of Lemma 30.

Lemma 37. *Let* $\xi \in \mathbb{P}^1$. *We have*

(i) $\dim \overline{\mathbf{B}}_{\mathbf{W}}(\Omega^0_u \times \Omega^0_{-u}, A_\infty) = 2$ *and* $\dim \overline{\mathbf{B}}_{\mathbf{W}}(\Omega^0_u \times \Omega^0_{-u}, A_\xi) = 1$ *if* $\xi \neq \infty$.

(ii) $\dim \overline{\mathbf{B}}_{\mathbf{W}}(\Omega^0_u \times \Omega^0_{-u}, B_\infty) = 1$ *and* $\dim \overline{\mathbf{B}}_{\mathbf{W}}(\Omega^0_u \times \Omega^0_{-u}, B_\xi) = 0$ *if* $\xi \neq \infty$.

11 Conclusion

From the classification, we can derive the following corollaries, some of which are used in [IM].

Corollary 1. *The primitive bilinear maps between modules of the class* $\mathcal{S}$ *are the following:*

(i) the Poisson products $P_{u,v}^{\delta_1,\delta_2}$*,*

(ii) the Poisson brackets $B_{u,v}^{\delta_1,\delta_2}$ *for* $\delta_1.\delta_2.(\delta_1+\delta_2)\neq 0$

(iii) the Lie brackets $B_{u,v}^{\delta_1,\delta_2}(\xi)$ *for* $\delta_1.\delta_2.(\delta_1+\delta_2)=0$ *and* $\xi\neq\infty$ *if* $\delta_1=\delta_2=0$*,*

(iv) Θ_∞*,*

(iv) the Grozman operation $G_{u,v}$*,*

(vi) $\eta(\xi_1,\xi_2,\xi_3)$ *for* ξ_1,ξ_2 *and* ξ_3 *are all distinct, and their* $\mathfrak{S}_3$*-symmetric*

(vii) the obvious map P^M*, and their* $\mathfrak{S}_3$*-symmetric.*

This follows easily by closed examination.

Corollary 2. *Let* M, N, $P\in\mathcal{S}^*$ *such that* $\mathbf{B}^0_{\mathbf{W}}(M\times N,P)\neq 0$*. Then the number of reducible modules among* M*,* N *and* P *is* 1 *or* 3*.*

Proof. Let $\pi\in\mathbf{B}^0_{\mathbf{W}}(M\times N,P)$ be non-zero. It follows from Theorem 1 that the three modules are reducible whenever Supp π is not one line. Otherwise, we can assume that. Assume Supp $\pi=V$. Then M, which admits a trivial quotient is reducible. Moreover, there is an almost-isomorphism $\phi:N\to P$, which proves that N and P are simultaneously reducible or irreducible. □

Corollary 3. *Let* M, N, $P\in\mathcal{S}^*$*. If* $\mathbf{B}_{\mathbf{W}}(M\times N,P)\neq 0$*, then we have*

$$(\deg M,\ \deg N,\ \deg P)\in\mathfrak{z}.$$

Proof. If $\mathcal{G}_{\mathbf{W}}(M\times N,P)\neq 0$, then the corollary follows from Theorem 2. Otherwise, we have $\mathbf{B}^0_{\mathbf{W}}(M\times N,P)\neq 0$. If $\{\deg M,\ \deg N,\ \deg P\}\subset\{0,1\}$, then $(\deg M,\ \deg N,\ \deg P)$ belongs to $\mathfrak{z}$. Otherwise, $\overline{\text{Supp}\ \pi}$ is one line for any non-zero $\pi\in\mathbf{B}^0_{\mathbf{W}}(M\times N,P)$. By the $\mathfrak{S}_3$-symmetry, we can assume that $\overline{\text{Supp}\ \pi}=V$. In such a case, N is isomorphic to P and we have $\deg M\in\{0,1\}$ and $\deg N=\deg P\notin\{0,1\}$. It follows that $(\deg M,\ \deg N,\ \deg P)$ belongs to $\mathfrak{z}$ as well. □

Let M, N, $P\in\mathcal{S}$. The triple (M,N,P) is called *mixing* if we have $\mathbf{B}^0_{\mathbf{W}}(M\times N,P)\neq 0$ and $\overline{\mathbf{B}}_{\mathbf{W}}(M\times N,P)\neq 0$. Here is a table of example of mixing triples:

Table 5: **Example of mixing triples** (M, N, P)

$M \times N$ or $N \times M$	P	A non-degenerate π_1	A degenerate π_2
$\Omega^0_u \times \Omega^1_0$	Ω^1_u	$P^{0,1}_{u,0}$	$(f, \alpha) \mapsto (\mathrm{Res}\,\alpha) df$
$\Omega^0_u \times \Omega^0_{-u}$	Ω^0_0	$P^{0,0}_{u,-u}$	$(f, g) \mapsto \mathrm{Res}\, f dg$

Corollary 4. *Any mixing triple (M, N, P) appears in the Table 5 and in each case π_1 and π_2 form a basis of $\mathbf{B_W}(M \times N, P)$.*

The corollary follows immediately from Tables 2 and 3.

Corollary 5. *For any triple (M, N, P) of indecomposable $\mathbf{W}$-modules in the class $\mathcal{S}$, we have $\dim \mathbf{B_W}(M \times N, P) \leq 2$.*

This follows easily from Theorem 1, Theorem 2 and the previous corollary. Note that the hypothesis that M, N and P are indecomposable is necessary. For example we have $\dim \mathbf{B_W}(X \times X, X) = 4$ if $X = \overline{A} \oplus \mathbb{C}$.

A Complete expression for $\mathbf{D}_{\delta_1,\delta_2,\gamma}(x, y)$

Recall that $\mathbf{D}_{\delta_1,\delta_2,\gamma} = \det \mathbf{M}$, where $\mathbf{M}$ is a 6×6 matrix whose entries are polynomials of degree 2 in x, y, δ_1, δ_2 and γ. The appendix provides the explicit formula for $\mathbf{D}$.

It is quite obvious that $\mathbf{D}_{\delta_1,\delta_2,\gamma}(x, y) = \mathbf{D}_{\delta_1,\delta_2,\gamma}(-x - 7, -y + 7)$. Thus, in order to provide more compact formulas, it is better to list the polynomials $\tilde{q}_{i,j}$ defined by

$$\mathbf{D}_{\delta_1,\delta_2,\gamma}(x - 7/2, y + 7/2) = C(\delta_1, \delta_2, \gamma) \sum_{i,j} \tilde{q}_{i,j}(\delta_1, \delta_2, \gamma) x^i y^j.$$

The polynomials $\tilde{q}_{i,j}$ are calculated with MAPLE. It turns out that the only non-zero polynomials are $\tilde{q}_{0,0}, \tilde{q}_{0,2}, \tilde{q}_{1,1}, \tilde{q}_{2,0}, \tilde{q}_{1,3}, \tilde{q}_{2,2}$ and $\tilde{q}_{3,1}$. It follows that $\mathbf{D}_{\delta_1,\delta_2,\gamma}$ is a polynomial of degree four in x and y and therefore $\tilde{q}_{1,3} = q_{1,3}$, $\tilde{q}_{3,1} = q_{3,1}$ and $\tilde{q}_{2,2} = q_{2,2}$, where the polynomials $q_{1,3}$, $q_{3,1}$ and $q_{2,2}$ are given in Section 7.3. The other non-zero polynomials $\tilde{q}_{i,j}$ are given by the following formulas:

$$
\begin{aligned}
16\tilde{q}_{0,0} =&[(4\delta_1+1)(4\delta_2+1)\gamma(1-\gamma)+16(\delta_1^2+\delta_2^2-\delta_1\delta_2)\delta_1\delta_2\\
&\quad+4(\delta_1^2+\delta_2^2-3\delta_1\delta_2(\delta_1+\delta_2))+(13(\delta_1^2+\delta_2^2)-50\delta_1\delta_2)+11(\delta_1+\delta_2)+2]\\
&\times[(4\delta_1+1)(4\delta_2+1)\gamma(1-\gamma)++16(\delta_1^2+\delta_2^2-\delta_1\delta_2)\delta_1\delta_2\\
&\quad+4(\delta_1^2+\delta_2^2)(\delta_1+\delta_2)-3(\delta_1^2+\delta_2^2+6\delta_1\delta_2)-7(\delta_1+\delta_2)+6],
\end{aligned}
$$

$$
\begin{aligned}
-4\tilde{q}_{0,2} =&(4\delta_1+1)^2\gamma^2(1-\gamma)^2\\
&+2(4\delta_1+1)((4\delta_1+1)\delta_2^2-2(4\delta_1+1)(\delta_1+1)\delta_2+4\delta_1^3+5\delta_1^2+2\delta_1+4)\gamma(1-\gamma)\\
&+(4\delta_1+1)^2\delta_2^4-4(4\delta_1+1)^2(\delta_1+1)\delta_2^3+(32\delta_1^4+112\delta_1^3+142\delta_1^2+52\delta_1-13)\delta_2^2\\
&-(64\delta_1^5+32\delta_1^4-92\delta_1^3+68\delta_1^2+82\delta_1-4)\delta_2\\
&-(4\delta_1+1)(\delta_1-1)(\delta_1+1)(\delta_1+2)(4\delta_1^2+\delta_1-6),
\end{aligned}
$$

$$
\begin{aligned}
-4\tilde{q}_{2,0} =&(4\delta_2+1)^2\gamma^2(1-\gamma)^2\\
&+2(4\delta_2+1)((4\delta_2+1)\delta_1^2-2(4\delta_2+1)(\delta_2+1)\delta_1+4\delta_2^3+5\delta_2^2+2\delta_2+4)\gamma(1-\gamma)\\
&+(4\delta_2+1)^2\delta_1^4-4(4\delta_2+1)^2(\delta_2+1)\delta_1^3+(32\delta_2^4+112\delta_2^3+142\delta_2^2+52\delta_2-13)\delta_1^2\\
&-(64\delta_2^5+32\delta_2^4-92\delta_2^3+68\delta_2^2+82\delta_2-4)\delta_1\\
&-(4\delta_2+1)(\delta_2-1)(\delta_2+1)(\delta_2+2)(4\delta_2^2+\delta_2-6),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\tilde{q}_{1,1} =&(28\delta_1^2\delta_2^2-4(\delta_1+\delta_2)\delta_1\delta_2+(\delta_1^2+\delta_2^2)-20\delta_1\delta_2-(\delta_1+\delta_2))\gamma(1-\gamma)\\
&\quad+4(7(\delta_1^2+\delta_2^2)-10\delta_1\delta_2)(\delta_1\delta_2)^2-4(\delta_1^3+\delta_2^3)\delta_1\delta_2\\
&\quad+(\delta_1^4+\delta_2^4-12(\delta_1^2+\delta_2^2)\delta_1\delta_2-6(\delta_1\delta_2)^2)\\
&\quad+2(\delta_1^2+\delta_1\delta_2+\delta_2^2)(\delta_1+\delta_2)-(\delta_1^2+\delta_2^2-14\delta_1\delta_2)-2(\delta_1+\delta_2).
\end{aligned}
$$

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